LINEAR ALGEBRAIC GROUPS: LECTURE 4

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1. An Abstract Approach

We can try to capture the concept of projective plane using points, lines and the incidence relation between them:

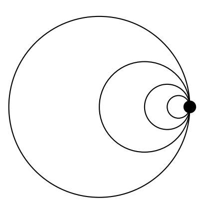
Definition. An abstract projective plane consists of a set P of points, a set L of lines, and an incidence relation $\mathrm{I} \subseteq P \times I$ (which we write as $p\mathrm{I}\ell$) which satisfy:

(1) For any two distinct points p and p', there exists a unique line ℓsuch that $p\mathrm{I}\ell$ and $p'\mathrm{I}\ell$;
(2) For any two distinct lines ℓ and ℓ', there exists a unique point p such that $p\mathrm{I}\ell$ and $p\mathrm{I}\ell'$;
(3) (Nondegeneracy axiom) There exist four points, no three of which lie on the same line.

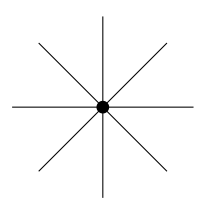
Remarkably, the nondegeneracy axiom is equivalent to the dual version where we switch the roles of lines and points: 'There exist four lines, no three of which contain the same point'. This axiom excludes precisely the following 7 types of degenerate example, which would otherwise count as abstract projective planes:

(1) The empty set.
(2) A single point.
(3) A line with no points.
(4) A line with more than four points:

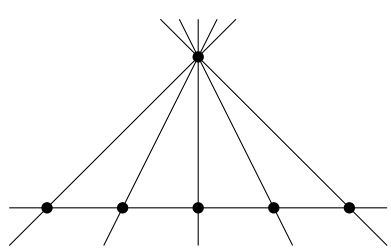
(5) Many lines with one point:



or



(6) A strange teepee:



(7) Yet another combination:

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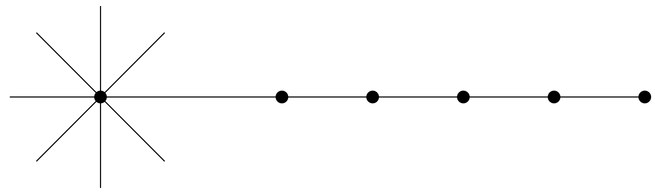


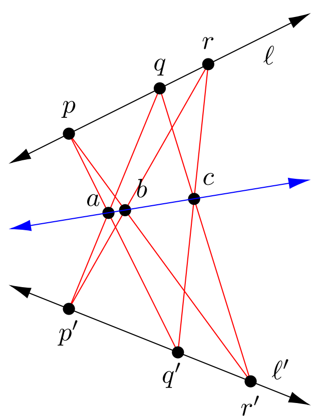
One might hope that with these excluded, every abstract projective plane would come from a field. As it turns out, there are many abstract projective planes that do *not* come from a field. However, we can add an axiom from Pappus to force these objects to be based on fields.

2. Pappus and the Projective Plane

Around 340 A.D., Pappus of Alexandria wrote about Euclid's *Porisms*, a treatise later lost in one of the great fires at the library of Alexandria. We don't actually even know precisely what Euclid meant by porism, but it's probably something similar to a theorem or proposition. Despite Pappus' antiquity, one of his theorems is important in projective geometry!

Theorem (Pappus's Hexagon Theorem)**.** *Let k be a field. If $p, q, r \in \ell$ and $p', q', r' \in \ell'$ are distinct points on distinct lines $\ell, \ell' \in kP^2$, then the points a, b and c all lie on the same line $m \in kP^2$.*

In the world of Euclidean plane geometry, the situation looks like this:



The red lines form the intersecting hexagon. It seems like such an ancient realization would have little value in trying to modernize geometry. In the last lecture we constructed abstract projective planes in a "modern" way, which need not arise from a field. However,

Theorem. *An abstract projective plane is isomorphic to the plane kP^2 for some field k if and only if Pappus's hexagon theorem holds in the abstract projective plane.*

Incidentally there is an analog for skew fields (such as the quaternions): an abstract projective plane comes from a skew field if and only if 'Desargues' theorem' holds.

3. Klein Geometry and Transitive Group Actions

In 1872, Felix Klein published his Erlangen Program. His major statement was that any kind of (highly symmetrical) geometry corresponds to a group G, the symmetry group of the geometry. Here, we can think of a geometry as a set of "figures" (points, lines, planes etc.), and its symmetry group is the group that maps like figures to like figures (points to points etc.).

In fact, we demand that each set of figures is something called a homogeneous set.

Definition. Given a group G, an action of G is a set X, together with a function

$$\begin{aligned} \alpha : G \times X \to \ & X \\ (g, x) \mapsto \ & \alpha(g, x) \ =: \ \alpha(g)(x), \end{aligned}$$

many times written just gx, if we are considering only one action of G, obeying

$$\begin{aligned} g(hx) = (gh)x \quad & \forall g, h \in G, \forall x \in X, \\ 1x = x. & \end{aligned}$$

We call this "an action of G on X," or a "G-set", or perhaps a "G-space".

Definition. An action of G on X is transitive if for all $x, y \in X$, there exists a $g \in G$ such that $gx = y$. A transitive G-set is also called a homogeneous G-space.

Example 1. Recall that the Euclidean group is $E(n) = SO(n) \ltimes \mathbb{R}^n$, where the $SO(n)$ component represents rotations and the $\mathbb{R}^n$ component represents translations. The collection

$$P \ = \ \{\text{set of points in } \mathbb{R}^n\}$$

is a homogeneous $E(n)$ space, as is the set of lines, L. Note that the choice of g need not be unique. For example, we could move the origin to the point $(1, 0)$ in $\mathbb{R}^2$ in a variety of ways. We could simply translate it one unit to the right; or we could translate it one unit up, followed by a $-90°$ rotation. Homogeneous spaces exist for all the geometries we have discussed so far:

- The elliptic plane, with group $G = SO(3)$;
- The hyperbolic plane, with group $G = SO(1, 2)$;
- Any projective space, with group $G = GL(n)$.

For a given G-set X, and a fixed element $x \in X$, we call the set $G_x := \{g \in G : gx = x\}$ the stabilizer of x.

Theorem. *If G is a group and X is a transitive G-space, then for all $x \in X$ there is an isomorphism (bijection)*

$$\begin{aligned} \phi : G/G_x \ & \to \ X. \\ [g] \ & \mapsto \ gx. \end{aligned}$$

Proof. **ϕ is well defined.** Suppose $[g] = [g']$ as equivalence classes in G/G_x. This implies

$$g' \ = \ gh$$

for some $h \in G_x$. But then

$$g'x \ = \ (gh)x \ = \ g(hx) \ = \ gx.$$

ϕ is one-to-one. If $g'x = gx$, then $g'x = ghx$ for any $h \in G_x$, so $[g'] = [g]$.

ϕ is onto. Given any $x' \in X$, transitivity implies there exists a $g \in G$ such that $gx = x'$. Hence $\phi[g] = x'$. $\square$

Notice that the empty set can be made into a G-set in exactly one way - vacuously, as there is no element to act on. Of course, there are more interesting examples.

Example 2. Consider the Euclidean plane with symmetry group $G = E(2)$. We can treat $\mathbb{R}^2$ as the plane $(x, y, 1)$ in $\mathbb{R}^3$, and can then write elements in $E(2)$ as matrices of the form

$$\left(\begin{array}{c|c} R & v \\ \hline 0\ 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ \hline 0 & 0 & 1 \end{array}\right),$$

with $\theta, a, b \in \mathbb{R}$. Now, let's look at the subgroup

$$H = \left\{ \left(\begin{array}{cc|c} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ \hline 0 & 0 & 1 \end{array}\right) : \theta \in \mathbb{R} \right\}.$$

This subgroup stabilizes (fixes) the origin, a single element in the homogeneous $E(2)$-space P, the collection of points in the Euclidean plane. Comparing to projective space, we have

$$\mathbb{R}^2 \ni (0,0) = p = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle \in \mathbb{R}P^2,$$

so $H \cong SO(2)$, and in Euclidean geometry we find the set of all points can be viewed as

$$P \cong G/H \cong E(2)/SO(2).$$

Example 3. Now, consider lines in the Euclidean plane. Just as rotations fix the origin (a point), translations parallel to the line fix the line. In $\mathbb{R}P^2$, we find that

$$H' = \left\{ \left(\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \in E(2) \right\}$$

stabilizes the x-axis, written as

$$\ell = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

Again, we find the set of all lines can be viewed as

$$L \cong E(2)/H'.$$

Due to our choice of the identity element in $SO(2)$, this collection L is actually the set of oriented lines.

4. Working Towards a Group-First approach

Since the basic theorem of Klein geometry is an isomorphism, we can also run the "Klein geometry machine" backwards. We could choose to begin with a subgroup $H \subseteq G$, and use it to determine a collection of figures, with G/H being isomorphic to that collection of figures. For example, we can define incidence relations between two types of figures to be the relations which are invariant under G-actions.

Definition. Given two types of figures, $X \cong G/H$ and $X' \cong G/H'$, a <u>relation</u> is a subset $R \subseteq X \times X'$. R is G-invariant if $(x, x') \in R$ implies $(gx, gx') \in R$.

As an example, in the Euclidean plane "p lies on ℓ" defines an $E(2)$-invariant relation $\mathrm{I} \in P \times L$, the incidence relation we have discussed previously.

Starting with a group G, we can generally work out all the subgroups. Then, we can work out all the G-invariant relations on the resulting transitive G-sets.