LINEAR ALGEBRAIC GROUPS: LECTURE 7

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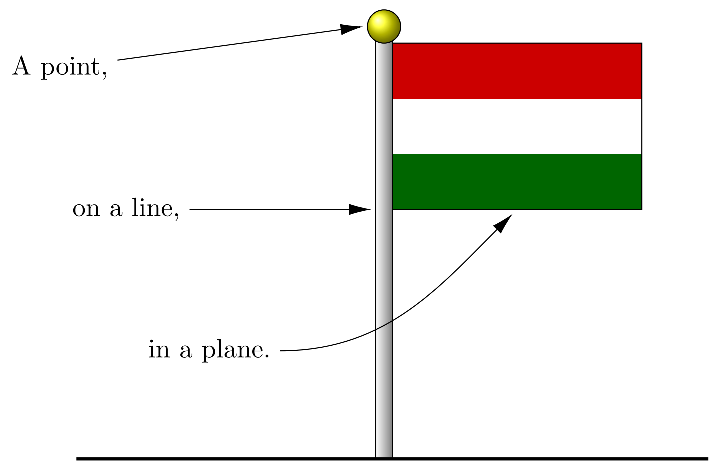
1. Parabolic Subgroups and Flag Varieties

So far, we've been studying Klein geometry, in particular projective geometry with symmetry group $GL(n)$. Here, the figures - points, lines, planes etc. - were various vector subspaces of k^n. However, it's actually important to consider fancier figures, assembled out of our basic figures. For example, we could consider a point on a line, or a point on a plane, or even a point on a line in a plane!

These figures are called flags, and the term probably stems from the fact that a typical flag on a flagpole can be considered as a point, on a line, in a plane.



We want to determine the space of flags of a particular type, and this will turn out to be a homogeneous space for some $H \subseteq GL(n)$, which will be called a parabolic subgroup.

Definition. Let the flag variety $F(n_1, n_2, \dots, n_\ell, n)$, where $n_1 < n_2 < \cdots < n_\ell < n$ are natural numbers, be the set of all flags of the form

$$V_{n_1} \subseteq V_{n_2} \subseteq \cdots \subseteq V_{n_\ell} \subseteq k^n,$$

where V_{n_i} is a linear subspace of k^n having dimension n_i for each i.

Example 1. $F(j, n) = \{j\text{-dimensional subspaces of } k^n\} = Gr(n, j)$, a Grassmannian.

Example 2. $F(1, 2, 3)$ consists of all $V_1 \subseteq V_2 \subseteq k^3$, where $\dim V_1 = 1$ and $\dim V_2 = 2$. In projective geometry, this is just $p \in \ell \subseteq kP^2$, where p is a point and ℓ is a line.

Example 3. A complete flag is something of the form

$$F_n \;:=\; F(1, 2, \dots, n-1, n).$$

This has elements of the form

$$V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq k^n,$$

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Having shown a few examples, let's now show

Theorem. *The flag variety $F(n_1, n_2, \ldots, n_\ell, n)$ is a homogeneous space for $GL(n)$, where $g \in GL(n)$ acts on the flag by*
$$V_{n_1} \subseteq V_{n_2} \subseteq \cdots \subseteq V_{n_\ell} \subseteq k^n \;\mapsto\; gV_{n_1} \subseteq gV_{n_2} \subseteq \cdots \subseteq gV_{n_\ell} \subseteq gk^n = k^n.$$

Proof. We need to show $GL(n)$ acts transitively on our flag. Let
$$V_{n_1} \subseteq V_{n_2} \subseteq \cdots \subseteq V_{n_\ell} \subseteq k^n,\; V'_{n_1} \subseteq V'_{n_2} \subseteq \cdots \subseteq V'_{n_\ell} \subseteq k^n \;\in\; F(n_1, n_2, \ldots, n_\ell, n).$$

Choose a basis $\{e_i\}_{i=1}^n$ for k^n, such that $\{e_i\}_{i=1}^{n_j} \in V_{n_j}$ for all j, so each collection $\{e_i\}_{i=1}^{n_j}$ forms a basis for each V_{n_j}. In a similar manner, choose a basis $\{e'_i\}_{i=1}^n$ for k^n, such that $\{e'_i\}_{i=1}^{n_j} \in V'_{n_j}$ for all j, so each collection $\{e'_i\}_{i=1}^{n_j}$ forms a basis for each V'_{n_j}. Through linear algebra, we know there exists a unique linear transformation such that $ge_i = e'_i$ for all i, so $gV_j = V'_j$ for all j, as desired. $\square$

As a homogeneous space, we know there exists a (parabolic) subgroup $P(n_1, n_2, \ldots, n_\ell, n) \subseteq GL(n)$ that fixes our "favorite flag" of this type, and by the basic Klein geometry theorem,
$$F(n_1, n_2, \ldots, n_\ell, n) \;\cong GL(n)/P(n_1, n_2, \ldots, n_\ell, n).$$

Example 4. The complete flag $F(1, 2, 3) \cong GL(3)/P(1, 2, 3)$, where $P(1, 2, 3)$ fixes our favorite flag, $V_1 \subseteq V_2 \subseteq k^3$, which can be written as
$$\langle (x_1, 0, 0) \rangle \;\subseteq\; \langle (x_1, x_2, 0) \rangle \;\subseteq\; k^3.$$

From last lecture, the subgroup that fixes V_1 is
$$P_{3,1} \;=\; \left\{ \begin{pmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & \ast & \ast \end{pmatrix} \in GL(3) \right\}.$$

Similarly, the subgroup that fixes V_2 is
$$P_{3,2} \;=\; \left\{ \begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \\ 0 & 0 & \ast \end{pmatrix} \in GL(3) \right\}.$$

The group that fixes both is then
$$P(1, 2, 3) \;=\; P_{3,1} \cap P_{3,2} \;=\; \left\{ \begin{pmatrix} \ast & \ast & \ast \\ 0 & \ast & \ast \\ 0 & 0 & \ast \end{pmatrix} \in GL(3) \right\},$$

which are the invertible upper triangular matrices.

Example 5. Consider instead $P(1, 3, 4)$. Choosing our favorite subspace approach, so that $V_1 = \langle (x_1, 0, 0, 0) \rangle$ and $V_3 = \langle (x_1, x_2, x_3, 0) \rangle$, we have that
$$P_{4,1} \;=\; \left\{ \begin{pmatrix} \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast \end{pmatrix} \in GL(4) \right\}$$

fixes V_1 and
$$P_{4,3} \;=\; \left\{ \begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast \end{pmatrix} \in GL(4) \right\}$$

fixes V_3. Thus, the subgroup that fixes the flag is the intersection,

$$P(1,3,4) \;=\; P_{4,1} \cap P_{4,3} \;=\; \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \in GL(4) \right\}.$$

These examples lead to a natural generalization.

Theorem. *The parabolic subgroups of $GL(n)$ are of the form*

$$P(n_1, n_2, \ldots, n_\ell, n) \;=\; \bigcap_{i=1}^{\ell} P_{n,n_i}.$$

As a question, why are they called parabolic? I'm not really sure, but there must have been a reason originally. **Stack Exchange** has a short question and answer available, but none of the commentary is necessarily enlightening.

Example 6. This time, we can look at $P(2,4,5)$. Again choosing our favorite subpaces (the natural embedding coordinates), we find

$$P_{5,2} \;=\; \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{pmatrix} \in GL(5) \right\}$$

fixes V_2 and

$$P_{5,4} \;=\; \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \in GL(5) \right\}$$

fixes V_4. Thus, the subgroup that fixes the flag is

$$P(2,4,5) \;=\; P_{5,2} \cap P_{5,4} \;=\; \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix} \in GL(5) \right\}.$$

2. Dimensionality and Finite Fields

Through the basic Klein geometry theorem, we know that

$$\dim\left(F(n_1, n_2, \ldots, n_\ell, n)\right) \;=\; \dim\left(GL(n)\right) - \dim\left(P(n_1, n_2, \ldots, n_\ell, n)\right).$$

This means that the dimension of the flag variety can be found by counting the number of zeros in the matrix form of the associated parabolic subgroup, as we already saw in the case of maximal parabolic subgroups.

In the case of finite fields, we can also work out the cardinality of $F(n_1, n_2, \ldots, n_\ell, n)$ over $\mathbb{F}_q$.

Example 7. Recall that $F_n = F(1, 2, \ldots, n-1, n)$ is called the complete flag variety. Then

$$F_n \;\cong\; GL(n)/B(n),$$

where $B(n)$ is the Borel subgroup defined as

$$B(n) \;=\; \bigcap_{i=1}^{n-1} P_{n,i} \;=\; \{\text{upper triangular matrices in } GL(n)\}\,.$$

For example,

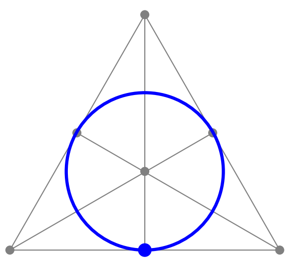
$$B(5) \;=\; \bigcap_{i=1}^{4} P_{5,i} \;=\; \left\{ \begin{pmatrix} \ast & \ast & \ast & \ast & \ast \\ 0 & \ast & \ast & \ast & \ast \\ 0 & 0 & \ast & \ast & \ast \\ 0 & 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & 0 & \ast \end{pmatrix} \in GL(5) \right\},$$

which cannot have zeros along the diagonal (as they are in $GL(5)$, and therefore invertible).

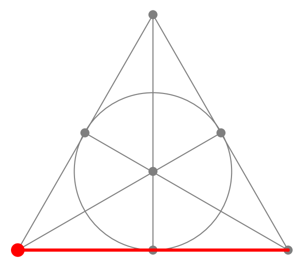
We worked with familiar ideas of points, lines, planes etc. in order to build complete flags. It turns out complete flags and Borel subgroups tell us all about the Geometry. However, we were working with the primary group $GL(n)$. If we instead work within $O(n)$, or the symplectic group, $Sp(n)$, or any other potential symmetry group, the Borel subgroups will be different. For our work in finite fields with $GL(n)$, however, we have a very nice result.

Theorem. *Over the finite field $\mathbb{F}_q$, $|F_n| = [n]_q!$.*

Example 8. If $n = 3$, then $F_3 = F(1, 2, 3) = \{p \in \ell \in kP^2\}$, a projective point in a projective line. When we choose $k = \mathbb{F}_2$, we again have the Fano plane. In this ambient space, a flag looks like a point in a line, such as



or



Now, the Fano plane has 7 points, each of which lie on exactly 3 lines. Through the beautiful duality, there are also 7 lines, each of which is incident with exactly 3 points. Either way, there are exactly $3 \cdot 7 = 21$ flags, so

$$|F_3| \;=\; 21.$$

If we were to instead apply the theorem, we would have that

$$\begin{aligned} |F_3| \;&=\; [3]_2! \\ &=\; [1]_2 \cdot [2]_2 \cdot [3]_2 \\ &=\; 1 \cdot (1+2) \cdot (1+2+2^2) \\ &=\; 21. \end{aligned}$$

Proof. In order to choose a complete flag $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq k^n$, we first choose a 1-dimensional subspace V_1 which is a point in kP^{n-1}. If $k = \mathbb{F}_q$, we know that

$$|kP^{n-1}| \;=\; [n]_q \;=\; 1 + \cdots + q^{n-1} \;=\; \frac{q^n - 1}{q-1}.$$

We then choose a 2-dimensional subspace V_2 with $V_1 \subseteq V_2$, which is equivalent to choosing a 1-dimensional subspace (or line) in

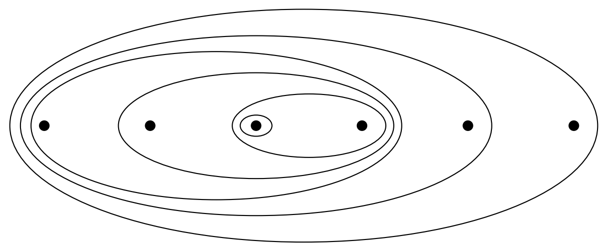
$$k^n/V_1 \;\cong\; k^{n-1},$$

so we are really picking a point in kP^{n-2}. As discussed previously, there are $[n-1]_q$ ways to do this. Continuing inductively, we arrive at our desired result. $\square$

We again have a strange, happy coincidence. If we were to let $q = 1$, relating to the nonexistent "field with one element" , then $[n]_q! = n!$. This means when we set $q = 1$, counting the number of flags means counting the number of possible nested subsets,

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{n-1} \subseteq n,$$

where n is a set with n elements, and S_i is a subset containing i elements.



A set theoretic complete flag on $n = 6$ elements