

POINTS AND MORPHISMS, FROM THE CLASSICAL AND SCHEME-THEORETIC POINT OF VIEW

BRIAN OSSERMAN

The aim of these notes is to give a concise introduction to the classical notions of points and morphisms for affine varieties (and more generally, algebraic sets) over a possibly non-algebraically closed field, and to explain how these notions may be expressed concisely in the language of schemes.

1. CLASSICAL AFFINE ALGEBRAIC SETS: THE ALGEBRAICALLY CLOSED CASE

Naively, given a field k and some $n \geq 0$, an affine algebraic set in k^n is the zero set of some finite collection of polynomial equations with n variables and coefficients in k . A morphism between two affine algebraic sets is a map defined by a tuple of polynomials. Armed with Hilbert's Nullstellensatz, it isn't too hard to make this precise in the case that k is algebraically closed, so we begin with this.

First, an **affine algebraic set** $V \subseteq k^n$ is a subset obtained as the common zero set of some $f_1, \dots, f_s \in k[x_1, \dots, x_n]$. Given a second affine algebraic set $W \subseteq k^m$, defined as the zero set of $g_1, \dots, g_t$, a **morphism** $V \rightarrow W$ is defined as follows. A m -tuple $(h_1, \dots, h_m)$ of polynomials in $k[x_1, \dots, x_n]$ defines a map

$$\vec{h} : k^n \rightarrow k^m$$

by sending $\vec{c} := (c_1, \dots, c_n)$ to $(h_1(\vec{c}), \dots, h_m(\vec{c}))$. A morphism is then a function $V \rightarrow W$ obtained as the restriction of some $\vec{h}$ as above.

Now, the above definitions are very naive and set-theoretic, but in the algebraically closed case, they turn out to behave well. Starting with the definition of algebraic set, one might wonder if we are losing information by only looking at the zero set, and not keeping track of the original defining polynomials. The answer is that we are losing very little information. Namely, it is easy to see that (irrespective of the algebraic closure hypothesis) V only depends on the ideal generated by the f_i , and in fact only on the radical of this ideal. So the best we could hope to do is to recover the radical of the ideal generated by the f_i from V , and Hilbert's Nullstellensatz says that in the algebraically closed case, we can do precisely that: if we simply take all polynomials which vanish on V , we will get precisely the radical of the ideal generated by the f_i . Thus, in working with algebraic sets over algebraically closed fields, it is irrelevant whether we think of them in terms of the sets themselves, or in terms of the radical ideals of polynomials used to define them.

Remark 1.1. We have glossed over one point: *a priori*, in going from finite sets of polynomials to arbitrary radical ideals, we could have introduced new ideals, if there were some ideals without finite generating sets. But the Hilbert basis theorem asserts that in fact, in a polynomial ring over a field, every ideal is finitely generated. Thus, this isn't an issue.

Before discussing morphisms, we make the following observations: given n, m and $\vec{h}$ as in the definition of morphisms, we obtain a ring homomorphism from $k[y_1, \dots, y_m]$ to $k[x_1, \dots, x_n]$ by sending each y_i to h_i , and extending k -linearly and multiplicatively. We denote this homomorphism by $\vec{h}^\sharp$. Then for any $(c_1, \dots, c_n) \in k^n$, and any $g \in k[y_1, \dots, y_m]$, we have the basic compatibility that

$$g(\vec{h}(c_1, \dots, c_n)) = (\vec{h}^\sharp(g))(c_1, \dots, c_n).$$

Now, for morphisms. We claim that here again, there is a correspondence between the set-theoretic definition, and a more algebraic approach. Given V, W as above, let $I \subseteq k[x_1, \dots, x_n]$ and $J \subseteq k[y_1, \dots, y_m]$ be the respective corresponding radical ideals. Given $\vec{h}$ as above, we claim that $\vec{h}$ gives rise to a morphism $V \rightarrow W$ if and only if $\vec{h}^\sharp$ maps J into I .

First suppose J is mapped into I . Given $(c_1, \dots, c_n) \in V$, we wish to see that $\vec{h}(c_1, \dots, c_n) \in W$. In fact, this is just a definition-chase which doesn't use the hypothesis that k is algebraically closed. Since W is the vanishing set of J , it suffices to see that for all $g \in J$, we have $g(\vec{h}(c_1, \dots, c_n)) = 0$. But by construction, we have

$$g(\vec{h}(c_1, \dots, c_n)) = (\vec{h}^\sharp(g))(c_1, \dots, c_n) = 0$$

since we have assumed $\vec{h}^\sharp(g) \in I$, and $(c_1, \dots, c_n) \in V$, which is the zero set of I . On the other hand, the converse uses the Nullstellensatz: suppose that $\vec{h}$ maps V into W . Given $g \in J$, we want to see that $\vec{h}^\sharp(g) \in I$. Given $(c_1, \dots, c_n) \in V$, since $\vec{h}(V) \subseteq W$, we have

$$\vec{h}^\sharp(g)(c_1, \dots, c_n) = g(\vec{h}(c_1, \dots, c_n)) = 0,$$

since $\vec{h}(c_1, \dots, c_n)$ is assumed in W , and g vanishes on W . Thus we see that $\vec{h}^\sharp(g)$ vanishes on V , and by the Nullstellensatz, it must be in I , as desired.

We thus see that in the algebraically closed case, we can make our definitions in terms of a naive set theory point of view, and they will still correspond very nicely with algebraic definitions.

2. CLASSICAL AFFINE ALGEBRAIC SETS OVER ARBITRARY FIELDS

If we wish to work over non-algebraically closed fields, the situation is more complicated, and it does not suffice to work naively with zero sets over the base field. Even if we are ultimately interested in the zero set over k , it is often helpful to consider first zero sets over extensions of k , which may be easier to analyze, and then attempt to use this to study the points over k . For instance, given a real algebraic set, the complex points are usually easier to understand, and the real points can be recovered as the fixed points of complex conjugation.

We introduce some convenient notation: given a subset $F \subset k[x_1, \dots, x_n]$, and k' extending k , denote by $Z_{k'}(F)$ the subset of $(k')^n$ consisting of simultaneous zeroes of all polynomials in F .

Example 2.1. Consider the elliptic curve defined (in the affine plane) by $y^2 = x^3 - x$. The $\mathbb{Q}$ -points of this are the solutions to this equation over $\mathbb{Q}$, and one can show that the only $\mathbb{Q}$ -points are $(0, 0)$, $(1, 0)$, and $(-1, 0)$. On the other hand, the $\mathbb{Q}(\sqrt{6})$ -points also contain the point $(2, \sqrt{6})$ (among others), the $\mathbb{R}$ -points contain an oval and a component going off to infinity, and the $\mathbb{C}$ -points look like a torus missing one point (the point at infinity). It is not hard to imagine a continuous involution of the torus with fixed points consisting of a disjoint pair of circles; this is how the complex and real points are related in this case.

We easily see that for non-algebraically closed k , just looking at the set $Z_k(I)$ for a radical ideal I may lose a great deal of information. For instance, if we set $k = \mathbb{R}$, and $I = (x_1^2 + x_2^2 + 1)$ or $I = (x_1^2 + x_2^2)$, then $Z_{\mathbb{R}}(I)$ is the empty set or a single point respectively. But $Z_{\mathbb{C}}(I)$ is infinite in both cases. That is, the (radical ideal generated by) the defining equations contains strictly more information than the zero sets. If, however, we are willing to work with zero sets over extensions of our base field as well, then we can recover the same algebraic information as before.

To develop the dictionary between algebraic and geometric definitions, we need a slightly more general statement of the Nullstellensatz than usual. There are many proofs of the Nullstellensatz, but for at least some of them, this more general statement is no harder – see Theorem 5.4 of [2].

Theorem 2.2 (Hilbert Nullstellensatz). *Let k be a field, and $\bar{k}$ its algebraic closure. Let $F \subseteq k[x_1, \dots, x_n]$ be a collection of polynomials, and $g \in k[x_1, \dots, x_n]$ any polynomial. Suppose that $g(c_1, \dots, c_n) = 0$ for all $(c_1, \dots, c_n) \in Z_{\bar{k}}(F)$. Then g is in the radical of the ideal generated by F .*

As a consequence, we see that for general k , we can think of our algebraic sets as being characterized in terms of their ideals, in terms of their zero sets over $\bar{k}$, or in terms of their zero sets over arbitrary extensions of k :

Corollary 2.3. *Let I, J be radical ideals of $k[x_1, \dots, x_n]$. Then the following are equivalent:*

- (1) $I = J$;
- (2) $Z_{\bar{k}}(I) = Z_{\bar{k}}(J)$;
- (3) $Z_{k'}(I) = Z_{k'}(J)$ for all field extensions k' of k .

Proof. It is clear that (1) implies (3) and (3) implies (2). The Nullstellensatz implies precisely that (2) implies (1). $\square$

Moreover, we can also characterize morphisms similarly, leading to a useful definition in the case that k is not algebraically closed:

Corollary 2.4. *Let $I \subseteq k[x_1, \dots, x_n]$ and $J \subseteq k[y_1, \dots, y_m]$ be radical ideals, and $\vec{h} = (h_1, \dots, h_n) \in k[y_1, \dots, y_m]^n$. Then the following are equivalent:*

- (1) $\vec{h}^\#(J) \subseteq I$;
- (2) for all $(c_1, \dots, c_n) \in Z_{\bar{k}}(I)$, we have $\vec{h}(c_1, \dots, c_n) \in Z_{\bar{k}}(J)$;
- (3) for all k' extending k , and all $(c_1, \dots, c_n) \in Z_{k'}(I)$, we have $\vec{h}(c_1, \dots, c_n) \in Z_{k'}(J)$.

The proof is the same as in the algebraically closed case, making use of the generalized Nullstellensatz statement.

We remark that when dealing with non-algebraically closed fields, it is important to keep track of what field a morphism is defined over.

Example 2.5. Consider the curves C_1 and C_2 , defined in the plane by the single equations $y^2 = x^3 + ax + b$ and $ny^2 = x^3 + ax + b$ respectively, where $a, b, n \in \mathbb{Q}$, with n non-zero.

If $n = m^2$ for some $m \in \mathbb{Q}$, we have the map (in fact an isomorphism) $f : C_1 \rightarrow C_2$ obtained by $(x, y) \mapsto (x, \frac{y}{m})$. Again, one can check both on points and on rings that this gives a map from C_1 to C_2 , and it is defined over $\mathbb{Q}$.

On the other hand, if n is not a perfect square in $\mathbb{Q}$, we see that as long as we consider C_1 and C_2 as curves over $\mathbb{Q}$, it is not possible to define the isomorphism f between them. On the other hand, if we consider them as curves over $\mathbb{Q}(\sqrt{n})$, we see that we can define f as before, and C_1 and C_2 are isomorphic. We say that f is **defined over** $\mathbb{Q}(\sqrt{n})$, but not over $\mathbb{Q}$.

In particular, f will give a natural bijection between the $\mathbb{Q}(\sqrt{n})$ points of C_1 and C_2 , but not the $\mathbb{Q}$ -points (which might look quite different).

Thus, we see that for both algebraic sets and morphisms, we can think of the fundamental data either algebraically (in terms of ideals, and homomorphisms sending ideals into one another) or geometrically (in terms of zero sets over extension fields, and polynomial maps sending zero sets into one another). To fix a convention, we will say that an **affine algebraic set** over k is described by a radical ideal $I \subseteq k[x_1, \dots, x_n]$, and a **morphism** between algebraic sets described by ideals $I \subseteq k[x_1, \dots, x_n]$ and $J \subseteq k[y_1, \dots, y_m]$ is described by the ring homomorphism $\vec{h}^\#$ sending J into I . Note that this is a bit sloppy, as different choices of $\vec{h}^\#$ might yield the same morphism on V , if the h_i used to define it differ by elements of I . We will address this issue shortly.

3. CLASSICAL AFFINE ALGEBRAIC SETS, ABSTRACTLY

What we have described above is, in essence, a category of imbedded affine algebraic sets over k . That is, our affine algebraic sets have thus far always come with an imbedding in a particular affine space, and this was used to define the concept of morphisms between them. We wish to show that from the algebraic point of view, this imbedding is in some sense extraneous data.

The first observations in this direction is the following: if we have an algebraic set V determined by a radical ideal $I \subseteq k[x_1, \dots, x_n]$, then we can define the **coordinate ring** $A(V) := k[x_1, \dots, x_n]/I$ of V . If also W is determined by $J \subseteq k[y_1, \dots, y_m]$, then we see that the condition on a morphism that $\vec{h}^\sharp$ map J into I is precisely equivalent to requiring that $\vec{h}^\sharp$ induce a ring homomorphism $A(W) \rightarrow A(V)$. This is promising, as it looks like we have now related the category of affine algebraic sets to the category of rings (more restrictively, rings which are finitely generated over k and do not have nilpotents).

However, the relationship is not quite so simple: not every ring homomorphism comes from a morphism of the corresponding algebraic sets. This is because we are starting off by assuming that $\vec{h}$ is “defined by polynomials,” which depends not only on the abstract rings, but also on the presentation of the rings as quotients of polynomial rings. For instance, in the trivial case that V and W are both just points, there is clearly only one morphism between them, but depending on the field k , there might be a lot of homomorphisms from k to itself! At first it may appear that this is a serious issue, but in fact there is a simple solution, coming from the observation that coordinate rings do not only have the structure of rings, but also the structure of k -algebras (that is, rings enriched with k -vector space structure, or equivalently, rings together with a homomorphism from k).

Proposition 3.1. *Given algebraic sets V, W determined by radical ideals $I \subseteq k[x_1, \dots, x_n]$ and $J \subseteq k[y_1, \dots, y_m]$, a homomorphism*

$$\varphi : A(W) \rightarrow A(V),$$

can be obtained as $\vec{h}^\sharp$ for some $(\vec{h}) \in k[x_1, \dots, x_n]^m$ if and only if φ is a homomorphism k -algebras.

Recall that a homomorphism of k -algebras is simply a ring homomorphism which is also k -linear.

Proof. By construction, any homomorphism obtained as $\vec{h}^\sharp$ is k -linear. Conversely, suppose that φ is k -linear. Then for $i = 1, \dots, m$ we may set h_i to be any lift to $k[x_1, \dots, x_n]$ of $\varphi(y_i)$, and it is clear that the resulting $\vec{h}^\sharp$ must agree with φ , since it is defined by extending the values on y_i using k -linearity and multiplicativity. $\square$

Thus, on the algebra side, we see that morphisms can in fact be defined without reference to the imbedding/presentation. All we have to do is work with coordinate rings considered as k -algebras. In categorical language, we say that the category of affine algebraic sets of k is *equivalent* to the category of finitely generated k -algebras without nilpotents.

It turns out the same approach can be applied to the points of an algebraic set. Namely, in our definition of $Z_{k'}(I)$, we use the presentation explicitly by extending the polynomial ring to k' and then taking the zero set of I in the larger ring. However, we observe that an extension k' of k is in particular a k -algebra, and this turns out to be a useful observation:

Proposition 3.2. *Given an algebraic set determined by a radical ideal $I \subseteq k[x_1, \dots, x_n]$, and any extension k' of k , the set $Z_{k'}(I)$ is in bijection with the set of k -algebra homomorphisms $A(V) \rightarrow k'$, via the map sending $(c_1, \dots, c_n) \in Z_{k'}(I)$ to the k -algebra homomorphism φ sending f to $f(c_1, \dots, c_n)$.*

Proof. First observe that the proposed map is well defined, since although f is only defined modulo I , we have $(c_1, \dots, c_n)$ in the vanishing set of I , so the value of f at $(c_1, \dots, c_n)$ does not depend

on the choice of representative. Thus, it suffices to produce an inverse map. Given a k -algebra homomorphism $\varphi : A(V) \rightarrow k'$, we observe the point $(\varphi(x_1), \dots, \varphi(x_n)) \in (k')^n$ in fact lies in $Z_{k'}(I)$, because I maps to 0 in $A(V)$ by definition. We claim this is the desired inverse. It is immediate from the construction that if we start with a point $(c_1, \dots, c_n)$, take the associated φ , and then consider $(\varphi(x_1), \dots, \varphi(x_n))$, we recover $(c_1, \dots, c_n)$. Conversely, given φ , and $f \in A(V)$, the fact that $f(\varphi(x_1), \dots, \varphi(x_n)) = \varphi f$ follows from the hypothesis that φ is a k -linear ring homomorphism. We thus obtain the desired correspondence. $\square$

Thus, the points can also be understood from an abstract algebraic point of view, without reference to the imbedding.

Remark 3.3. The distinction between k -algebra homomorphisms and arbitrary ring homomorphisms is not to be taken lightly. In 1964, Serre ([3]) produced examples of projective complex varieties whose equations are related to one another by applying an automorphism of $\mathbb{C}$, but which have different fundamental groups, and in particular are not even homeomorphic to one another in the classical complex topology. For a more recent survey and further work, see also [1].

4. AFFINE ALGEBRAIC SETS FROM A SCHEME POINT OF VIEW

From here, it is easy enough to see how to relate schemes to the classical point of view of algebraic sets and their morphisms. Before discussing the “right” way to understand it, we explore the naive point of view, which looks pretty bleak. In the context of points, even if k is algebraically closed, $\text{Spec } A(V)$ contains not only the classical points of V , but also extra generic points. If k is not algebraically closed, the situation is even worse, as $\text{Spec } A(V)$ contains points which are related to but not in bijection with classical points over algebraic extensions of k .

Example 4.1. Consider the case of $\mathbb{A}_k^1 = \text{Spec } k[t]$. The closed points correspond to maximal ideals of $k[t]$, which is a PID, so we see that we get a closed point for every monic irreducible polynomial $p(t) \in k[t]$.

In particular, for any k -point of $\mathbb{A}_k^1$, which is simply some $c \in k$, we have the irreducible polynomial $t - c$, so the k -points are naturally contained among the closed points of $\mathbb{A}_k^1$.

On the other hand, if k is not algebraically closed, we also have irreducible polynomials of higher degree. We can relate these to points of $\mathbb{A}_{\bar{k}}^1$ by considering the roots of the polynomials $p(t)$. If $c \in \bar{k}$ is any element of the algebraic closure, we could consider it as corresponding to the closed point given by $p(t)$, the minimal polynomial of c over k . On the other hand, given any $p(t)$, there is some $c \in \bar{k}$ a root of $p(t)$. But this is not a unique correspondence, so we can summarize as follows:

There is a map from $\bar{k}$ -points of $\mathbb{A}_k^1$ to closed points of $\mathbb{A}_k^1$ which is surjective, and such that any $c, c' \in \bar{k}$ map to the same point if and only if they have the same irreducible polynomial over k .

The situation with morphisms is at least as bad, and directly comparable to what we discussed above in the context of rings.

Example 4.2. Consider morphisms from the single point $\text{Spec } k$ to itself. Classically, there should only be one morphism from a point to a point: the identity map. We see that this is not the case with schemes. Such morphisms correspond to ring homomorphisms (in the opposite direction) from k to itself k . The possibilities for these depend on the field k , but they certainly include $\text{Aut}(k)$, which could be extremely large (if $k = \bar{\mathbb{Q}}$, then $\text{Aut}(k) = \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$).

On the other hand, if $k = k'(t)$ for some smaller field k' and t a transcendental element, then we have the non-surjective map corresponding to $t \mapsto t^2$.

Which is stranger? That there can be many maps from a point to itself, or that there exist non-invertible maps from a point to itself?

What can I say, other than “it’s not a bug, it’s a feature!”

Allow me to explain. First, the fact that we see $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, an object of great interest to number-theorists, popping up in the context of schemes, is no coincidence, but rather fits into Grothendieck’s vision of using scheme theory to unify number theory and algebraic geometry. Similarly, the map $t \mapsto t^2$ mentioned above is nothing more than the map induced on generic points by the map $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ given by $t \mapsto t^2$. The fact that the map isn’t invertible is an encapsulation of the fact that it comes from a morphism of the affine line which certainly isn’t invertible.

We now explain how the classical notions of points and morphisms can be expressed in scheme land. As mentioned above, if V is an affine algebraic set over k with coordinate ring $A(V)$, then $A(V)$ naturally carries the structure of a k -algebra, which is to say, it comes with a homomorphism $k \rightarrow A(V)$. In scheme world, this means that we are given not only the scheme $\text{Spec } A(V)$, but also a morphism $\text{Spec } A(V) \rightarrow \text{Spec } k$, which is called the **structure morphism**. Similarly, if k' is a field extension of k , we obtain a morphism of schemes $\text{Spec } k' \rightarrow \text{Spec } k$. Since morphisms of affine schemes correspond precisely to ring homomorphisms in the other direction, it is simple to reformulate our earlier discussion in scheme terms. Specifically, we have:

Proposition 4.3. *Let V be an affine algebraic set over k . For any field k' extending k , the classical k' -points $Z_{k'}(V)$ can be described in scheme language as the set of morphisms $\text{Spec } k' \rightarrow \text{Spec } A(V)$ which commute with the given maps to $\text{Spec } k$.*

Proof. Proposition 3.2 asserts that the k' -points of V are in bijection with k -algebra homomorphisms $A(V)$ to k' , and it is elementary to verify that these in turn are the same as ring homomorphisms $A(V) \rightarrow k'$ which commute with the given inclusions of k into each. The proposition then follows from the correspondence between ring homomorphisms and affine scheme morphisms. $\square$

For morphisms, the story is the same:

Proposition 4.4. *Let V, W be affine algebraic sets over k . Then morphisms (as algebraic sets) from V to W are in natural bijection with scheme morphisms $\text{Spec } A(V) \rightarrow \text{Spec } A(W)$ commuting with the structure morphisms to $\text{Spec } k$.*

Proof. We have seen in Proposition 3.1 that morphisms from V to W are in bijection with k -algebra homomorphisms $A(W)$ to $A(V)$, so the proposition again follows from the correspondence between ring homomorphisms and affine scheme morphisms. $\square$

Moral. We see that the correct way to think of an algebraic set over k in scheme terms is not as an individual scheme, but as a scheme together with a structure morphism to $\text{Spec } k$. Accordingly, many classical properties of varieties, when translated into the scheme setting, will be phrased as properties of morphisms rather than of individual schemes.

In a similar vein, the underlying set/topological space of a scheme should really be viewed as a technical tool rather than an intuitive geometric object. Indeed, even for something as basic as the classical notion of points, we have now seen that they are best understood in terms of morphisms rather than in terms of the underlying set of the scheme.

REFERENCES

1. Robert Easton and Ravi Vakil, *Absolute Galois acts faithfully on the components of the moduli space of surfaces: a Belyi-type theorem in higher dimension*, International Mathematics Research Notices **2007** (2007), no. 20, 1–10.
2. Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, 1986.
3. Jean-Pierre Serre, *Exemples de variétés projectives conjuguées non homéomorphes*, Comptes Rendus de l’Académie des Sciences. Série I. Mathématique **258** (1964), 4194–4196.