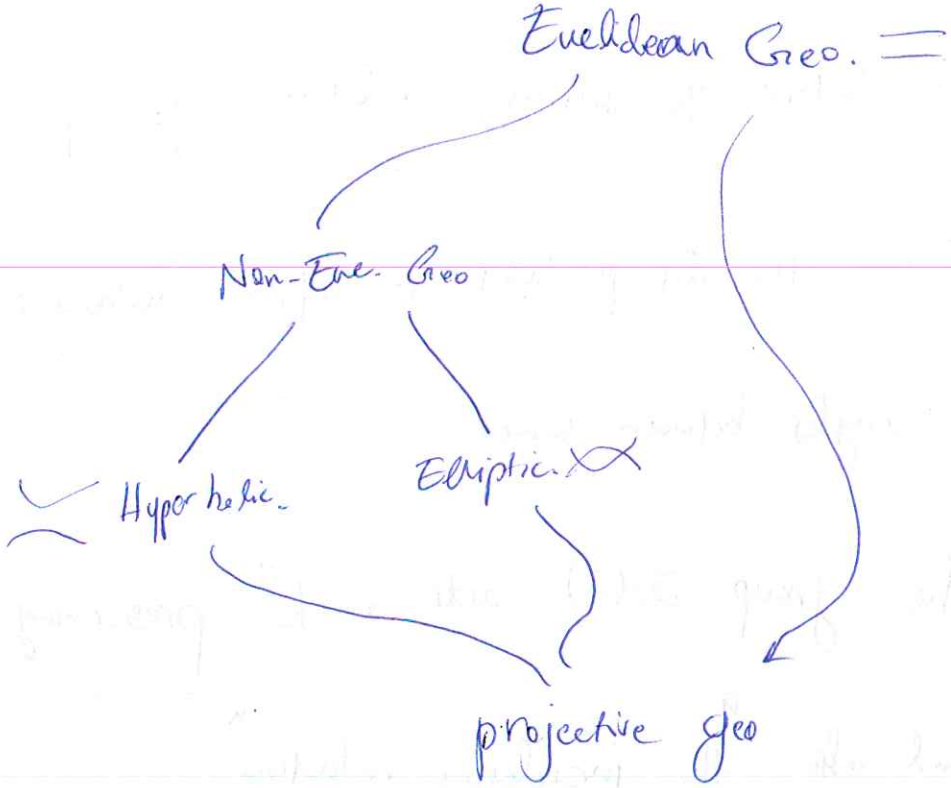
Session 2



Geometries & their algebraic groups

1) Elliptic Geometry.

Take k^3 ($\mathbb{R}^3$) with the usuall inner product: $v.w = \sum v_i w_i$.

From the sphere $X = \{ v \in k^3 : v.v = 1 \}$

and defines a set P of points and L of lines as follows:

$P = \{ 1d \text{ subspaces of } k^3 \}$

$L = \{ 2d \text{ subspaces of } k^3 \}$

and define the incidence relation: "point p lines on line ℓ" $\equiv p \subseteq \ell$

* We use the dot product to define distances between points & angles between lines.

The group $SO(3)$ acts on k^3 preserving the dot product and also "the incidence relation"

$$p \subseteq \ell \Rightarrow gp \subseteq g\ell \quad \forall g \in SO(3)$$

2) Hyperbolic Geometry

Give k^3 the "Lorentzian dot product"

$$v.w = -v_1 w_1 - v_2 v_2 + v_3 w_3$$

so that:

$$X = \{ v \in k^3 : v.v = 1 \}$$

and

$P = \{$ 1d subspaces of k^3 with non-empty intersection with $X \}$

$L = \{$ 2d " " " " $\}$

Let

$O(1,2) = \{ g \in GL(3) : gv \cdot gw = v \cdot w$ (Lorentzian dot product) $\forall v, w \in k^3$

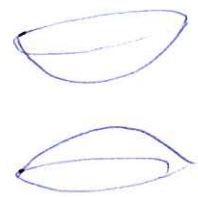
$SO(1,2)$ preserves the dot product and incidence relation.

3) Euclidean Geometry:

Take k^3 with the degenerate dot product:

$v \cdot w = v_3 w_3$

To compare:

	dot product	
elliptic:	+ + +	
Euclidean:	0 0 +	
hyperbolic:	− − +	

Now $X = \{ v \in k^3 : v \cdot v = 1 \}$

$P = \{$ 1d subspaces of k^3 having non-empty intersection w/ $X \}$

$L = \{$ 2d " " " " $\}$

What about the symmetry group?

The Euclidean group: $E(3) = \left\{ \left(\begin{array}{c|c} R & v \\ \hline 0 & 1 \end{array} \right) : \begin{array}{l} R \in SO(2) \\ \& \ v \in k^2 \end{array} \right\}$

which preserves X :

$$\left(\begin{array}{c|c} R & v \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} x \\ y \\ \pm 1 \end{pmatrix} = \begin{pmatrix} R\begin{pmatrix} x \\ y \end{pmatrix} + v \\ \pm 1 \end{pmatrix}$$

[can we write $E(3)$ as a quotient group?]

and the distances and angles.

So the main weird difference with this case in that $E(3)$ is not the group of all $\det = 1$ transformations preserving the degenerat det product.

And Now:

4) Projective Geometry

points & lines:

$P = \{ 1d \text{ subspaces of } k^3 \}$

$L = \{ 2d \quad " \quad " \}$

We won't define distances and angles, but we define the incidence relation: $p \subseteq \ell$.

Now the symmetry group is $GL(3)$.

* Projective geometry goes back at least to the Renaissance painters, who developed perspective.

If we fix any plane $X \subseteq k^3$ not containing the origin most (but not all) 1d subspaces of k^3 will intersect X in a single point.

Thus most points $p \in P$ correspond to points of X.

The remaining points of P are called "points at infinity"

Doing this for any dimension, we get projective $(n-1)$-space.

$$kP^{n-1} = \{ 1d \text{ subspaces of } k^n \}$$

Most points in kP^{n-1} will intersect the plane (for example)

$X = \{ (x_1, \ldots, x_{n-1}, 1) : x_i \in k \}$ in a single point.

So we get a 1-1 correspondence between X and an "open dense" (what is the topology? Zariski) set in kP^{n-1}.

Thus points and Lines in 3 other geometries are points and lines in projective geometry, and their symmetry group is a subgroup of $GL(3)$.

$\boxed{\text{Session 3}}$

I) Projective Geometry

Recall: for any field k:

$$kP^n = \{\text{1d subspaces of } k^{n+1}\}$$

<u>Thm.</u> As sets:

$$kP^n \cong k^n + k^{n-1} + \dots + k^0$$

isomorphic

$k^0 = \{0\}$

means disjoint union

These pieces $k^n, k^{n-1}, \dots$ are called <u>Schubert Cells</u>

& this is a Schubert decomposition

<u>Proof:</u> Any 1d subspace of k^{n+1} has the form

$p = \langle (x_1, \dots, x_{n+1}) \rangle$ w. $(x_1, \dots, x_{n+1}) \neq 0$ where $\langle\ \rangle$

means span.

if $x_{n+1} \neq 0$

$$p = \langle (\frac{x_1}{x_{n+1}}, \dots, \frac{x_n}{x_{n+1}}, 1) \rangle := \langle (y_1, y_2, \dots, y_n, 1) \rangle$$

a description which is unique.

So we get a bijection between k^n & the set of all 1d subspaces of k^{n+1} of the form $\langle (x_1, \dots, x_{n+1}) \rangle$ with $x_{n+1} \neq 0$.

if x_{n+1} then p is really a 1d subspace of k^n

$k^n \cong \{(x_1, \dots, x_n, 0) : x_i \in k\}$. So we get the bijection:

$$kP^n \cong k^n + kP^{n-1}$$

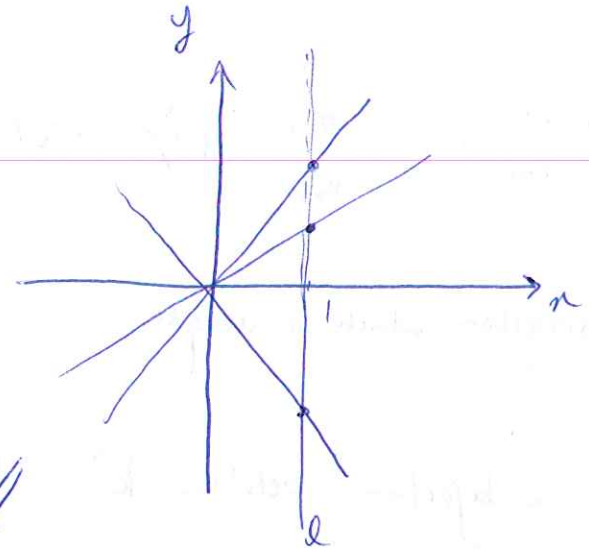
by induction:

$$kP^n \cong k^n + k^{n-1} + \dots + k^0$$

Examples:

1) $kP^1 \cong k^1 + k^0$

line ℓ point at ∞



1-1 correspondence between lines with slope $\neq \infty$ and

point on line $\ell \cong k^1$

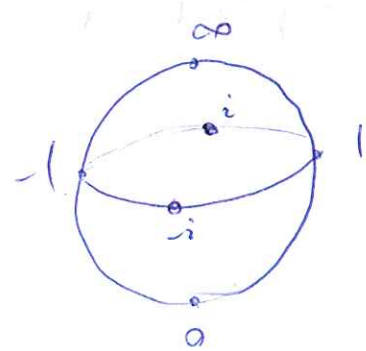
line with slope $= \infty \cong k^0$

2) $\mathbb{R}P^1$ as a set is $\mathbb{R} + \{\infty\}$. But as a topological space it's S^1,

the 1-point compactification of $\mathbb{R}$.

3) $\mathbb{C}P^1 \cong \mathbb{C} + \{\infty\}$. As a space this is S^2, or as a complex variety!!

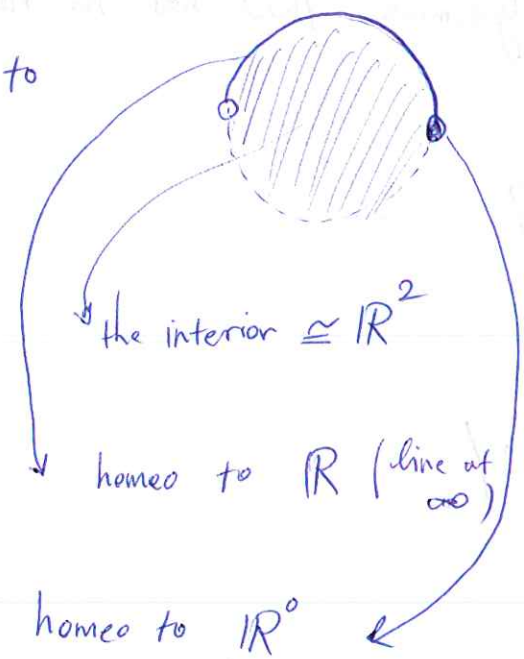
the Riemann sphere.



4) $\mathbb{RP}^2$ is homeomorphic to $S^2/\{v \sim -v\}$

or to a disc D^2 with $v \sim -v$ on boundary.

or to



$$\Rightarrow \mathbb{RP}^2 \cong \mathbb{R}^2 + \underbrace{\mathbb{R} + \{\infty\}}_{\mathbb{RP}^1}$$

5) Any finite field k has q elements where q is a prime power:

$q = p^m$ where p is prime and $m \in \mathbb{N}$.

Moreover all fields with q elements are isomorphic, so we

write $\mathbb{F}_q$ for "the" field with q elements.

$\mathbb{F}_p$ for p prime is just $\mathbb{Z}/p\mathbb{Z}$ with usual $+$.

To get $\mathbb{F}_{p^m}$ with $n>1$ we can take $\mathbb{F}_p$ & throw in all m roots of some degree-m polynomial that has no roots in $\mathbb{F}_p$.

What's the cardinality of $\mathbb{F}_q P^n$?

$$|\mathbb{F}_q P^n| = |\mathbb{F}_q^n + \mathbb{F}_q^{n-1} + \cdots + \mathbb{F}_q^0|$$

$$= |\mathbb{F}_q^n| + |\mathbb{F}_q^{n-1}| + \cdots$$

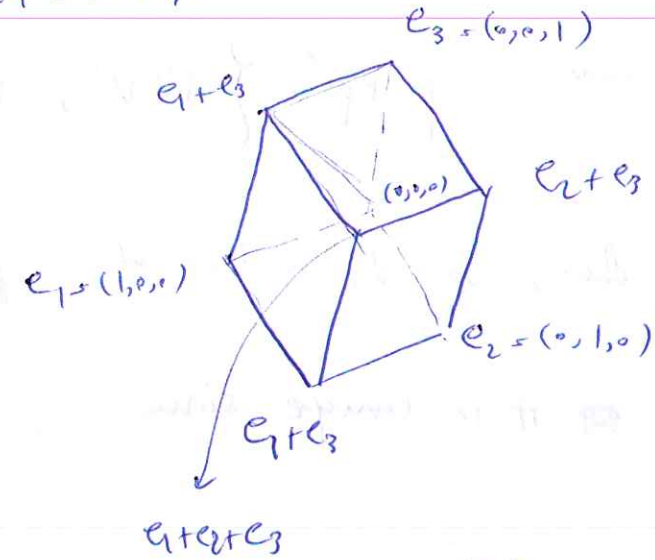
$$= |\mathbb{F}_q|^n + |\mathbb{F}_q|^{n-1} + \cdots$$

$$= q^n + q^{n-1} + \cdots + 1$$

$$= \frac{q^{n+1}-1}{q-1}$$

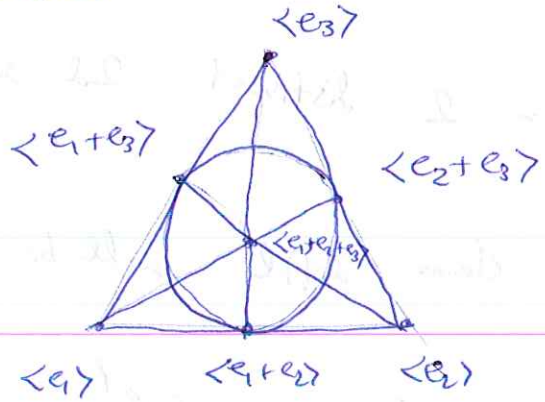
which is called the q-integer: $[n+1]_q$ (it approaches $n+1$ as $q \to 0$)

6) $\mathbb{F}_2 P^2$ is called the "Fano plane", the smallest projective plane. $|\mathbb{F}_2 P^2| = 2^2+2+1=7$



1d subspaces of $\mathbb{F}_2^3$?

lines in the following figure.



For any two points, passes a line.

any two lines intersect at one point.

<u>Thm.</u> In any projective plane kP^2:

1) for any 2 distinct p and p', $\exists!$ line ℓ w. $p, p' \in \ell$

2) For any 2 distinct ℓ, ℓ', $\exists!$ point p w. $p \subseteq \ell, \ell'$

Proof

1) Given 2 distinct 1d subspaces of k^3, p & p' the

subspace sum $p + p' = \{ v + v', \ v \in p, v' \in p' \}$

is 2 dim, so it's a line w. $p, p \subseteq \ell$.

and it is unique since ... .

2) Given 2 distinct 2d subspaces $\ell, \ell' \subseteq k^3$

we claim $\ell \cap \ell'$ will be a 1d subspace $p = \ell \cap \ell'$, and

the clearly $p \subseteq \ell, \ell'$.

$\dim(\ell + \ell') = \dim(\ell) + \dim(\ell') - \dim(\ell \cap \ell')$

...

I Axiomatic Projective Geometry

Def. An abstract projective plane consists of a set of points P, a set of lines L, and an incidence relation $I \subseteq P \times L$. if $(p, \ell) \subseteq I$ we say $pI\ell$ [p lies on ℓ]

Now we demand:

1) For any distinct $p, p' \in P$ $\exists!$ $\ell \in L$ w. $pI\ell$, $p'I\ell$

2) For any distinct $\ell, \ell' \in P$ $\exists!$ $p \in P$ w. $pI\ell$ & $pI\ell'$

3) Non-degeneracy axiom:

V1. There exist 4 point, no 3 of which lie on the same line.

V2. " " 4 lines " " contain the same point.

The two versions are equivalent.

The non-degeneracy eliminates:

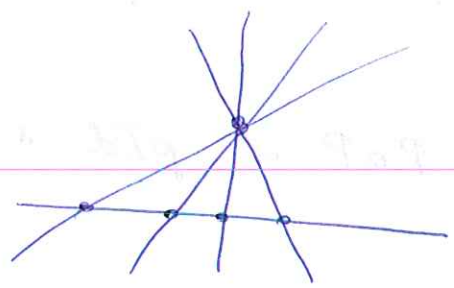
1) $P = L = \phi$

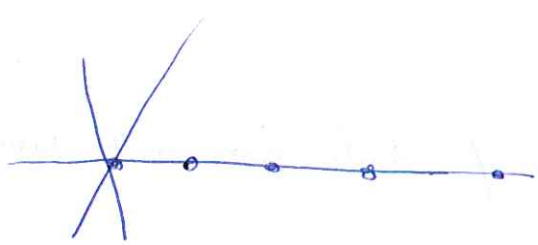
2)

3)

4) 

5) 

6) 

7) 

with these axioms, do projective ~~fo~~ planes come from a field?

Not ..

Session 4

Last time, "Schubert cell" should have been "Bruhat cell".

A Bruhat cell is isomorphic to k^n; the corresponding schubert cell is the closure of the Bruhat cell.

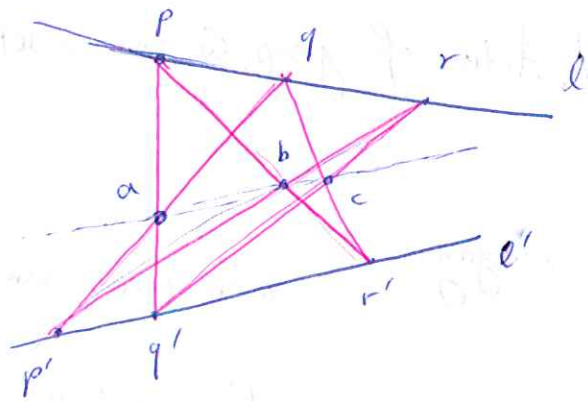
So if $k=\mathbb{R}$, Bruhat cells are open n-balls & schubert cells are closed n-balls.

I Projective Planes & Axiomatic Projective geometry

* Around 400 AD, Pappus wrote about Euclid's "Porisms".

<u>Thm.</u> Pappus's Hexagon theorem: if k is any field and we have this configuration of points & lines in kP^2:

then points a, b and c lie on a line.



Proof: See wikipedia

In fact:

<u>Thm.</u> An abstract projective plane is isomorphic to the plane kP^2 coming from some field iff Pappus's hexagon theorem holds in this abstract projective plane.

I Klein Geometry

Any kind of (highly symmetrical) geometry corresponds to a group G (the symmetry group of the geometry).

There will be various sets of "figures" (e.g. points, lines, circles, triangles, ...) and G acts on each of these sets. In fact we demand each set is a "homogenous space" for G.

<u>Def.</u> Action of group G on a set X: $\alpha: G \times X \longrightarrow X$

$$(g, x) \longmapsto \alpha(g,x) =: \alpha(g)x =: gx$$

obeying:

* Associative law $g(hx) = (gh)x \quad \forall g, h \in G, \forall x \in X$
* Identity law $1x = x \quad \forall x \in X$

We call this an action of G on X, or a <u>G-set</u>, or a <u>G-space</u>.

<u>Def.</u> An action of G on X is transitive if $\forall x,y \in X$, $\exists g \in G : gx = y$

A transitive <u>G-set</u> is called a <u>homogeneous G-space</u>.

<u>Example:</u> if $G = E(n)$ (the Euclidean group) the the set of points $P = \mathbb{R}^n$ in a homogeneous G-space, as is the set of lines L.

The same holds for points & lines in the other geometries we've discussed:

elliptic $G = SO(3)$

hyperbolic $G = SO(1,2)$

projective $G = GL(n)$

<u>Thm.</u> if G is a group & X is a transitive G-set, for any $x \in X$ there is an isomorphism (a bijection) $\phi : G/G_x \longrightarrow X$

where G is the stabilizer of $x \in X$: $G_x = \{g \in G : gx = x\}$.

and $\phi([g]) = gx$

Proof: 1) ϕ is well defined: if $[g'] = [g]$ $\left(g' = gh \text{ for some } h \in G_x \right)$

$\phi([g']) = \phi([g])$ since $g'x = ghx = gx$

2) and also 1-1: if $g'x = g(x) \Rightarrow [g'] = [g]$

3) and also onto: given any $x' \in X$, transitivity implies $\exists y \in G$

s.t. $gx = x' \Rightarrow \phi([g]) = x'$

Note: the empty set always can be made into a G-set in one way, and this action is transitive. But $\phi \neq G/H$ (for some H).

So did the above theorem go wrong? No. Since $\nexists x \in X$

Example: 1) Euclidean plane: $G = E(2)$

the subgroup $H = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

stabilizes a point, namely $p = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

Here $H \cong SO(2)$. So in Euclidean geometry:

$$P = E(2)/SO(2) \rightsquigarrow \text{really } H$$

2) $G = E(2)$, translations along a line fix that line.

So let $H' = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$

It stabilizes: $\ell = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

but so does the 180° rotation $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

So $G_\ell = \langle H', 180^\circ \text{ rotation} \rangle \Rightarrow E(2)/G_\ell \cong L$ [the space of all lines]

also G/H' would be the space of oriented lines

In Klein geometry, we pick a group G and say each subgroup $H \subseteq G$ determines a "type of figure", with G/H being the set of figures of that type.

We can define incidence relations between two types of figures to be G-invariant relations.

Given 2 types of figures: $X = G/H$

$X' = G/H'$

a relation between these is a subset $R \subseteq X \times X'$.

R is G-invariant if:

$$(x, x') \in R \Rightarrow (gx, gx') \in R \quad \forall g \in G$$

E.g. in Euclidean geometry "p lies on ℓ" defines an $E(2)$-invariant relation: $I \subseteq P \times L$.

* Starting from a group, we can workout all its subgroups & all the invariant relations on the resulting transitive G-sets.

Lecture 5

Projective Geometry from a Kleinian viewpoint

According to Klein, a group gives a geometry.

So let's try: $G = GL(n)$

Different types of geometrical figure in projective geometry correspond to different subgroups of $GL(n)$.

Here are some fundamental kind of figures:

Def. Let the Grassmannian $Gr(n,j)$ $(1 \leq j \leq n-1)$ be the set of all j-dimentional linear subspaces of k^n.

Examples:

$$Gr(n,1) = kP^{n-1} \overset{!}{=} \{\text{points of } kP^{n-1}\}$$

$$Gr(n,2) = \{\text{lines in } kP^{n-1}\}$$

$$G(n,3) = \{\text{planes in } kP^{n-1}\}$$

$$\vdots$$

$$Gr(n,j) = \{(j-1)\text{-planes in } kP^{n-1}\}$$

$$\vdots$$

$$Gr(n,n-1) = \{\text{hyperplanes in } kP^{n-1}\}$$

$GL(n)$ acts on each $Gr(n,j)$ via:

$$gL = \{ gv : v \in L \} \qquad L \in Gr(n,j)$$

& they all are homogeneous spaces of $GL(n)$:

any $L \in Gr(n,j)$ has a basis $v_1, \dots, v_j \in k^n$ & similarly

$L' \in Gr(n,j)$ has basis $v_1', \dots, v_j' \in L'$

and we can find $g \in GL(n)$ s.t. $gv_i = v_i'$ so that $gL = L'$

thus by our theorem last time $Gr(n,j) = GL(n)/P_{n,j}$

where $P_{n,j}$ is the subgroup that fixes a chosen $L \in Gr(n,j)$

The subgroups $P_{n,j}$ are "<u>maximal parabolic</u>" subgroups of $GL(n)$.

Indeed, any (linear) algebraic group G will have maximal parabolic subgroups that fix the "nicest" types of figures in its geometry.

* To study $P_{n,j}$ choose a nice j-dim subspace of k^n:

$$L = \{ (x_1, \dots, x_j, 0, 0, \dots, 0) \in k^n \}$$

& define:

$$P_{n,j} = \{ g \in GL(n) : gL = L \}$$

What's it like?

<u>Examples.</u>

$P_{3,1}$ = subgroup of $GL(3)$ that fixes a point in the projective plane.

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix}$$

this matrices fix $L = P_{3,1}$

any vector $\in L$ looks like this, where $*$ means an arbitrary element of the field,

$P_{3,2}$ = subgroup of $GL(3)$ that fixes a line in the projective plane.

$$\underbrace{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}}_{= P_{3,2}} \begin{pmatrix} * \\ * \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ 0 \end{pmatrix}$$

In fact $P_{3,2} \cong P_{3,1}$ as a group but they are not conjugate in $GL(3)$.

$P_{4,1}$:

$$\begin{array}{c} 4-1 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \underbrace{}_{1} \end{array} \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{pmatrix} \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$P_{4,2}$:

$$\begin{array}{c} 4-2 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \underbrace{}_{2} \end{array} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ 0 \\ 0 \end{pmatrix}$$

$P_{4,3}$

$$\begin{array}{c} 4-3 \left\{ \begin{array}{c} \\ \\ \\ \end{array} \right. \\ \underbrace{}_{3} \end{array} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix} = \begin{pmatrix} * \\ * \\ * \\ 0 \end{pmatrix}$$

<u>Thm.</u>

$$P_{n,j} = \left\{ \begin{array}{c} j\left\{ \begin{array}{c} \\ \\ \end{array} \right. \\ n-j\left\{ \begin{array}{c} \\ \\ \end{array} \right. \end{array} \begin{array}{c} \left(\begin{array}{c|c} X & Y \\ \hline 0 & Z \end{array} \right) \\ \begin{array}{cc} \underbrace{}_{j} & \underbrace{}_{n-j} \end{array} \end{array} : \begin{array}{l} X, Y, Z \text{ are arbitrary} \\ \text{matrices of the correct shape} \\ \& \ \det(X)\det(Z) \neq 0 \end{array} \right\}$$

* if $k = \mathbb{R}$, any linear algebraic group G is a manifold, and if $H \subseteq G$ is algebraic as well then G/H is also a manifold & $\dim(G/H) = \dim G - \dim H$

* For any field, an (linear) algebraic group G is an (affine) algebraic variety, and if $H \subseteq G$ is algebraic as well, then G/H is also an algebraic variety (not necessarily affine) &

$$\dim(G/H) = \dim(G) - \dim(H)$$

<u>Thm.</u> $\dim(Gr(n,j)) = j(n-j)$

from the form of $P_{n,j}$ $\dim(P_{n,j}) = n^2 - (n-j)j$

and since $Gr(n,j) \cong GL(n)/P_{n,j}$ $\dim(Gr(n,j)) = (n-j)j$

* $\dim(Gr(n,j)) = \dim(Gr(n,n-j))$ & in fact $Gr(n,j) \cong Gr(n,n-j)$

since using an inner product we get a 1-1 & onto ~~map~~ map

$L \longmapsto L^{\perp}$ which is called duality in projective space.

a Psedo-Pascal's triangle for $d_{n,j} := \dim(Gr(n,j))$

But this is just the multiplication table!!

$d_{1,0} = 1$

$d_{2,0} = 2 \quad d_{2,1} = 2$

$d_{3,0} = 3 \quad d_{3,1} = 4 \quad d_{3,2} = 3$

$d_{4,0} = 4 \quad d_{4,1} = 6 \quad \cdots$

$\vdots$

* Pascal's triangle shows up when we count the number of points in $Gr(n,j)$ when $k = \mathbb{F}_q$. We'll get the "q-deformed" Pascal's triangle.

Remember $kP^n \cong k^n + k^{n-1} + \cdots + k^0$

so if $k = \mathbb{F}_q$: $\quad |kP^n| = \dfrac{q^{n+1}-1}{q-1} := [n+1]_q$ (called n+1-th q-integer)

Also: $kP^{n-1} = Gr(n,1)$. How does $|Gr(n,1)| = [n]_q$ generalize to other Grassmannians?

Def. The q-factorial $[n]_q!$ is given by:

$$[n]_q! = [n]_q [n-1]_q \cdots [1]_q$$

and the <u>q-binomial coefficient</u> $\binom{n}{j}_q$ is given by:

$$\binom{n}{j}_q = \frac{[n]_q!}{[j]_q!\,[n-j]_q!}$$

Thm. if $k = \mathbb{F}_q$: $|Gr(n,j)| = \binom{n}{j}_q$

Note: $\binom{n}{j}$ counts the number of j-element subsets of an n-element set, while $\binom{n}{j}_q$ counts the number of j-dimentional subspaces of a n-dim vector space over $\mathbb{F}_q$.

So in some mysterious way a vector space over "the field with one element" $(q=1)$ is just a finite set!!