10/19 (Diversity & Excellence in Math Conference later today & tomorrow)

continuing: Chain complexes from simplicial abelian groups

<u>Thm</u> - there's a functor (equivalence)

$$C : Ab^{\Delta^{op}} \xrightarrow{\sim} Ch(Ab)$$

given as follows:

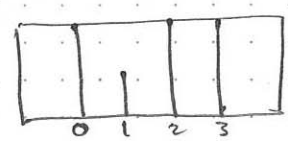
- on objects $X : \Delta^{op} \to Ab$ define

$$C(X) \in Ch(Ab)$$

by $C(X)_n = X([n])$ (group of n-simplices)

where $\partial : C(X)_{n+1} \to C(X)_n$

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i \quad (\partial_i = X(d_i))$$

with $\partial^2 = 0$. $(d_1 =$  $)$

- on morphisms

$$\begin{tikzcd} \Delta^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\alpha\Downarrow"] & Ab \end{tikzcd}$$

define

$$C(\alpha) : C(X) \to C(Y)$$

ie

$$C(\alpha)_n : \underset{\substack{\| \\ X([n])}}{C(X)_n} \to \underset{\substack{\| \\ Y([n])}}{C(Y)_n}$$

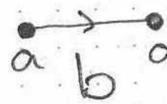
to be precisely $\alpha_{[n]}$

(it all just works in category theory - front-load difficulty with levels of abstraction)

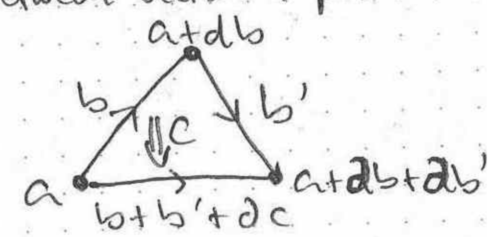
moreover, C is an <u>equivalence</u>,
i.e. there is a functor $D: Ch(Ab) \to Ab^{\Delta^{op}}$
such that $D \circ C \cong 1_{Ab^{\Delta^{op}}}$ (natural isomorphisms)
$C \circ D \cong 1_{Ch(Ab)}$

<u>Prf</u> (sketch) — (hard part is unpacking ∂)
C is clearly a functor: $C(\beta \circ \alpha) = C(\beta) \circ C(\alpha)$
because $C(\beta \circ \alpha)_n = (\beta \circ \alpha)_{[n]} = \beta_{[n]} \circ \alpha_{[n]}$ (by defn of nat. trans.)

to get $D: Ch(Ab) \to Ab^{\Delta^{op}}$, we need to see how given $A \in Ch(Ab)$ we get $D(A) \in Ab^{\Delta^{op}}$.
— abelian group of 0-simplices $= A_0$ $\quad \bullet\, a \in A_0$
— abelian group of 1-simplices



$b \in A_1$ st $\partial b = a' - a$
ie $= A_0 \oplus A_1$ (expecting undergrads to understand natural iso between vectors + points)
— abelian group of 2-simplices
$c \in A_2$ st $\quad A_0 \oplus A_1^2 \oplus A_2$



need to check:
$a + \partial(b + b' + \partial c) = a + \partial b + \partial b'$
yes! because $\partial^2 = 0$ (Pascal's Δ)
— and so on....

then, check $C(D(A)) \cong A$ via a natural iso
and that $D(C(X)) \cong X$
(Dold-Kan theorem)

given a chain complex C, since $\partial^2 = 0$, note that $\mathrm{im}(\partial : C_{n+1} \to C_n)$ "boundaries"

$$\cap$$

$\ker(\partial : C_n \to C_{n-1})$ "cycles"

so we can form <u>homology groups</u>, abelian groups

$$H_n(C) = \frac{\ker(\partial : C_n \to C_{n-1})}{\mathrm{im}(\partial : C_{n+1} \to C_n)}$$

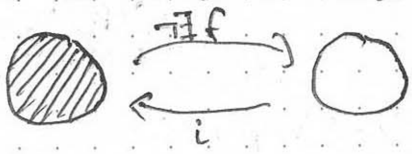
we can write $H(C) = \{H(C)_n\}_{n=0}^{\infty}$, so $H(C)$ is a $\mathbb{N}$-graded abelian group, or $H(C) \in \mathrm{Ab}^{\mathbb{N}}$.

<u>Thm</u> $H : \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Ab}^{\mathbb{N}}$ is a functor.

(Emmy Noether noticed all this great stuff)

↳ Betti numbers don't tell you what to do with maps

(algtop shows you some "[illegible]" apps)



$fi = 1_{S^1}$

on H_1: $\{0\} \overset{?}{\rightleftarrows} \mathbb{Z}$ $\quad \neg\exists\, H_1(f) \circ H_1(i) = 1_{\mathbb{Z}}$

Q&A: now we'll turn monads into simplicial objects, then do homology!

but there's a subtlety: monads are functors

(walking monad) $M \longrightarrow K$ (2-category)

(Δ_a)

but need Δ^{op}