10/24 The Bar Construction

(the coolest thing in the universe]

<u>Thm</u> given adjoint functors $\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, "U"] \end{tikzcd}$
and an object $d \in D$,
there's a simplicial object

$$\bar{d} \in D^{\Delta^{op}}$$

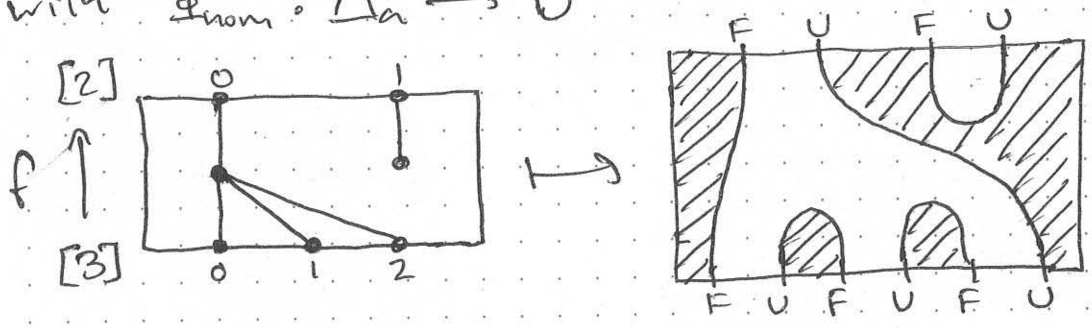
given by the <u>bar construction</u>

$$\bar{d} : \Delta^{op} \xhookrightarrow{\iota^{op}} \Delta_a^{op} \xrightarrow{\Phi_{hom}} D^D \xrightarrow{ev_d} D \qquad (*)$$

<u>Prf</u> (trivial; explain functors) the first is the opposite of the inclusion $\Delta \hookrightarrow \Delta_a$.
recall that an adjunction in Cat uniquely determines a 2-functor
(walking comonad) $W \xrightarrow{\Phi} Cat$

$$\Phi_{obj} :: * \mapsto D$$

with $\Phi_{hom} : \Delta_a^{op} \longrightarrow D^D$



finally, there's an <u>evaluation functor</u>

$$ev_d : D^D \longrightarrow D$$

(naturality gives functoriality)

$$\begin{tikzcd} F \arrow[d, Rightarrow, "\alpha"'] \arrow[r, mapsto] & F(d) \arrow[d, "\alpha_d"] \\ G \arrow[r, mapsto] & G(d) \end{tikzcd}$$

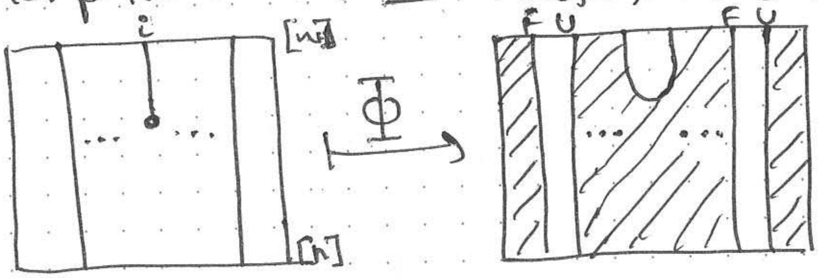
now let's think about the whole composite.

what does $\bar{d} : \Delta^{op} \to D$ really do?

$$
\begin{aligned}
[1] &\longmapsto FU(d) \\
[2] &\longmapsto FUFU(d) \\
[3] &\longmapsto FUFUFU(d) \\
&\vdots \\
[n] &\longmapsto (FU)^n(d)
\end{aligned}
$$

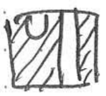
and on morphisms in Δ $\quad d_0, \dots, d_n : [n] \to [n+1]$

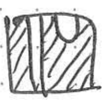
in Δ^{op},

$d_i =$ 

you get morphisms in D

$$\big(\overbrace{FU \cdots FU \cdots FU}^{n}\big)(d) \xrightarrow{1_{(FU)^{n-i-1}}\, \varepsilon_{FU(d)}\, 1_{(FU)^i}} (FU)^{n-1}(d)$$

 $FUFU(d) \xrightarrow{\varepsilon_{FU(d)}} FU(d)$ (generated from)

 $FUFU(d) \xrightarrow{FU(\varepsilon_d)} FU(d)$

<u>example</u>: a G-set is a set X equipped with an action such that

$$A : G \times X \to X \qquad g_1(g_2 x) = (g_1 g_2) x$$
$$(g, x) \mapsto gx \qquad 1x = x$$

there's a category of G-sets, Set^G.

there's an adjunction $\quad Set \underset{U}{\overset{F}{\rightleftarrows}} Set^G$ $\quad(\perp)$

where $F(X) = G \times X$

which is trivially a G-set $\quad g_1(g_2, x) = (g_1 g_2, x)$.

given a G-set X, the bar construction gives a simplicial G-set $\overline{X}$ as follows:

$$\begin{tikzcd}
\cdots & FUFUFU(X) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, equal] & FUFU(X) \arrow[r, shift left, "\varepsilon_{FU(X)}"] \arrow[r, shift right, "FU(\varepsilon_X)"'] \arrow[d, equal] & FU(X) \arrow[d, equal] \\
& G\times G\times G\times X & G\times G\times X & G\times X
\end{tikzcd}$$

G-set of 2-simplices $\quad$ G-set of 1-simplices $\quad$ G-set of 0-simplices

(where if S is a G-set, the counit

$$\begin{tikzcd}
\varepsilon_S : FU(S) \arrow[r] \arrow[d, equal] & S \\
G\times S \arrow[ur, "A_S"'] &
\end{tikzcd}$$

the action!)

so given a 1-simplex in $\overline{X}$, what are its two vertices?

(g_1, g_2x) $\quad \underset{0}{\bullet} \xrightarrow{(g_1, g_2, x)} \underset{1}{\bullet} \quad$ (g_1g_2, x)

0th face map $(\varepsilon_{FU(X)})$: $(g_1, g_2, x) \mapsto (g_1g_2, x)$

1st face map $(FU(\varepsilon_X))$: $(g_1, g_2, x) \mapsto (g_1, g_2x)$

in $\overline{X}$ the equation $g_1(g_2x) = (g_1g_2)x$ is being realized as a rewrite.

the bar construction decomposes an algebraic structure & then reassembles it, replacing equations with 1-simplices and higher ones representing "equations between equations", etc!