10/26 Bar Construction (continued)

– for G-sets:

dfn: a <u>groupoid</u> is a category where every morphism is an isomorphism, ie has an inverse.

$$1_x \circlearrowleft x \underset{f^{-1}}{\overset{f}{\rightleftarrows}} y \circlearrowright 1_y$$

dfn: a <u>monoid</u> is a category with one object.

(if $* \in M$, $\hom(*,*)$ has associative/unital comp.)

$$\text{true} \circlearrowleft \bullet \circlearrowright \text{false}$$ with "and" or "or" as comp

dfn: a <u>group</u> is a groupoid that's a monoid.

$$0 \circlearrowleft \bullet \circlearrowright 1$$ with "+" – $\mathbb{Z}_2$

(in nature/history, groups are symmetries of an object)

*realize the abstract "•" as something concrete through (functorial) <u>action</u>:

dfn: given a group G, the <u>category of G-sets</u> is Set^G:

- objects are functors $G \xrightarrow{A} Set$,

ie a set $A(*)$

and a function $A(g): A(*) \to A(*)$ $\forall g \in G$

st $A(gh) = A(g) \circ A(h)$, $A(1_*) = 1_{A(*)}$

• morphisms are natural transformations

$$\begin{tikzcd} G \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathrm{Set} \end{tikzcd}$$

ie a function $\alpha_*: A(*) \to B(*)$

obeying naturality

$$\begin{tikzcd} A(*) \arrow[r] \arrow[d, "A(g)"'] & B(*) \arrow[d, "B(g)"] \\ A(*) \arrow[r] & B(*) \end{tikzcd} \quad \forall g \in G$$

($\exists$ nice correspondence between symmetries of the octahedron and the cube)

— this idea puts group theory into a larger context.

<u>Thm</u>: there are adjoint functors

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Set}^G \arrow[l, bend left, "U"] \end{tikzcd}$$

where U maps any $A: G \to \mathrm{Set}$ to its <u>underlying set</u> $A(*)$ and any map of G-sets $\alpha: A \Rightarrow B$ to its <u>underlying function</u> $\alpha_*: A(*) \to B(*)$

and F maps any set X to the G-set A whose underlying set $A(*) = G(*,*) \times X$ and where group elements act as follows

$$A(g): G \times X \to G \times X$$
$$(h, x) \mapsto (gh, x)$$

and given $f: X \to Y$, $F(f): F(X) \to F(Y)$ is

$$G \times X \to G \times Y$$
$$(g, x) \mapsto (g, f(x))$$

<u>Prf</u> (sketch) to show F is the left adjoint of U, need a natural isomorphism:

$$G\text{-Set}(F(X), A) \xrightarrow[\sim]{\alpha} \text{Set}(X, U(A))$$

(let's write it simpler -)

$$\underset{\text{shorthand}}{\text{Set}^G}(G \times X, \underset{A(*) \text{ as G-Set}}{Y}) \xrightarrow[\sim]{\alpha} \text{Set}(X, \underset{\text{as set}}{Y})$$

given a map of G-sets $f: G \times X \to Y$,
we'll choose this function $\alpha(f): X \to Y$
$x \mapsto f(1, x)$

conversely, given a function $f: X \to Y$
we have a map of G-sets $\alpha^{-1}(f): G \times X \to Y$
(this is really a map, because $(g, x) \mapsto g \cdot f(x)$

$\alpha^{-1}(f)(h(g,x)) = \alpha^{-1}(f)(hg, x) = (hg) f(x)$

$h \cdot \alpha^{-1}(f)(g,x) = h(g(f(x)))$)

- need to check naturality. ∎

$\boxed{\text{HW}}$: determine η/ε of this adjunction.

(already used ε for bar construction)

<u>Q&A</u>: what's special about G to give adjunction?

* one-objectness ~ monoids work so far too,

$\Rightarrow$ but will use bar to get <u>classifying space</u> of G

(simplicial set $\xrightarrow{|-|}$ topological space $\xrightarrow{\pi_1}$ back to G)

generalizations: monoids have functor $1 \to M$,

in general $F: C \to D$ gives $\text{Set}^D \xrightarrow{F^*} \text{Set}^C$ which has a left adj.