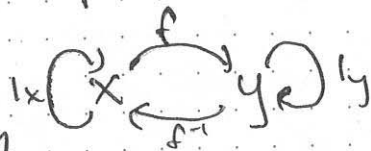


10/26 Bar Construction (continued)

- for G -sets:

dfn: a groupoid is a category where every morphism is an isomorphism, i.e. has an inverse.

dfn: a monoid is a category with one object.



(if $* \in M$, $\text{hom}(*, *)$ has associative/unital comp.)

true  false

with "and" or "or" as comp

dfn: a group is a groupoid that's a monoid.



with "+" - \mathbb{Z}_2

(in nature/history, groups are symmetries of an object)

*realize the abstract " \bullet " as something concrete through (functorial) action:

dfn: given a group G , the category of G -sets is Set^G :

• objects are functors $G \xrightarrow{A} \text{Set}$,

i.e. a set $A(*)$

and a function $A(g): A(*) \rightarrow A(*) \quad \forall g \in G$

st $A(gh) = A(g) \circ A(h)$, $A(1_*) = 1_{A(*)}$

• morphisms are natural transformations

$$G \begin{array}{c} \xrightarrow{A} \\ \Downarrow \alpha \\ \xrightarrow{B} \end{array} \text{Set}$$

ie a function $\alpha_*: A(*) \rightarrow B(*)$

obeying naturality $\begin{array}{ccc} A(s) & \downarrow & B(s) \\ A(*) & \xrightarrow{\alpha_*} & B(*) \end{array} \quad \forall g \in G$

(Nice correspondence between symmetries of the octahedron and the cube)

— this idea puts group theory into a larger context.

Thm: there are adjoint functors

$$\text{Set} \begin{array}{c} \xleftarrow{U} \\ \perp \\ \xrightarrow{F} \end{array} \text{Set}^G$$

where U maps any $A: G \rightarrow \text{Set}$ to its underlying set $A(*)$ and any map of G -sets

$\alpha: A \Rightarrow B$ to its underlying function $\alpha_*: A(*) \rightarrow B(*)$

and F maps any set X to the G -set A whose underlying set $A(*) = G(*, *) \times X$ and where group elements act as follows

$$\begin{aligned} \text{Alg} : G \times A &\rightarrow G \times A \\ (g, x) &\mapsto (gh, x) \end{aligned}$$

and given $f: X \rightarrow Y$, $F(f): F(X) \rightarrow F(Y)$ is

$$\begin{aligned} G \times X &\rightarrow G \times Y \\ (g, x) &\mapsto (g, f(x)) \end{aligned}$$

Prf (sketch) to show F is the left adjoint of U , need a natural isomorphism:

$$G\text{-Set}(F(X), A) \xrightarrow[\cong]{\alpha} \text{Set}(X, U(A))$$

(let's write it simpler -)

$$\text{Set}^{G(\text{short hand } A(f) \text{ as } G\text{-set})}(G \times X, Y) \xrightarrow[\cong]{\alpha} \text{Set}^{(\text{as set})}(X, Y)$$

given a map of G -sets $f: G \times X \rightarrow Y$,

we'll choose this function $\alpha(f): X \rightarrow Y$

$$x \mapsto f(1, x)$$

conversely, given a function $f: X \rightarrow Y$

we have a map of G -sets $\alpha^{-1}(f): G \times X \rightarrow Y$

(this is really a map, because

$$\alpha^{-1}(f)(h(g, x)) = \alpha^{-1}(f)(hg, x) = (hg)f(x)$$

$$h \circ \alpha^{-1}(f)(g, x) = h(g(f(x)))$$

$$(g, x) \mapsto g \cdot f(x)$$

— need to check naturality. ▣

HW: determine η/ϵ of this adjunction.

(already used ϵ for bar construction)

Q&A: what's special about G to give adjunction?

* one-objectness ~ monoids work so far too,

→ ^{but} will use bar to get classifying space of G

(simplicial set $\xrightarrow{\epsilon}$ topological space $\xrightarrow{\pi_1}$ back to G)

generalizations: monoids have functor $1 \rightarrow M$,

in general $(F: C \rightarrow D)$ gives $\text{Set}^D \xrightarrow{F_*} \text{Set}^C$ which has a left adj.