10/29: Bar Construction for Groups

given any group \( G \), we have an adjunction

\[
\text{Set} \xrightarrow{\mathcal{F}} \text{Set}^G
\]

given any \( G \)-set \( X \), the counit \( \varepsilon_X : \mathcal{F}(X) \to X \)

sends \((g, x) \mapsto gx\). (This is the free map).

the bar construction gives a simplicial \( G \)-set

\[
\tilde{X} = \mathcal{F} \mathcal{U} \mathcal{F} \mathcal{U} \mathcal{F} \mathcal{U}(X) \Rightarrow \mathcal{F} \mathcal{U}(X)
\]

2-simplices \( 1 \)-simplices \( 0 \)-simplices

(as discussed previously), here the maps are

built using the counit, more concretely,

\[
\tilde{X}(n[,]) = (\mathcal{F} \mathcal{U})^n(X)
\]

and

\[
\tilde{X}(d_i) : (\mathcal{F} \mathcal{U})^{n+1}(X) \to (\mathcal{F} \mathcal{U})^n(X)
\]

\[
= (\mathcal{F} \mathcal{U})^{n+1} \varepsilon \mathcal{U} : G^{n+1} \times X \to G^n \times X
\]

\[
(g_0, \ldots, g_n, x) \mapsto (g_0, \ldots, g_{i-1}, g_i \cdot g_{i+1}, \ldots, g_n, x)
\]

or \((g_0, \ldots, g_n, g_{n+1})\).

for example,

\[
(g_0, g_1, g_2, x)
\]

\[
(g_0, g_1, g_2, x) \quad (g_0, g_1, g_2, x) \quad (g_0, g_1, g_2, x)
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\[
(g_0, g_1, g_2, x) \quad (g_0, g_1, g_2, x)
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\]

each element of \( X \) gives many vertices of \( \tilde{X} \) connected by edges. all vertices above correspond to some element of \( X \):

\[
g_0(g, g_{2x}) = g_0g_1(g_2x) = (g_0, g_1, g_2, x)
\]
the bar construction realizes proofs
(the edges are the proofs of equality)
— this is a crucial idea of modern mathematics.

Our simplicial G-set \( X \) has an underlying simplicial set \( U(X) : \Delta^{op} \rightarrow \text{Set}^G \rightarrow \text{Set} \).

Then we can turn any simplicial set into a topological space:

Thus: there's a functor called geometric realization

\[ 1 : \text{Set}^{\Delta^{op}} \rightarrow \text{Top} \]

Prf: (sketch) — “visually evident”

To prove it, there’s a functor (category theorists could talk about this diagram for days)

we can turn any simplex into a space;
this functor extends to geometric realization using the Yoneda embedding,

\[ \Delta \rightarrow \text{Set}^{\Delta^{op}} \]

\[ \text{Top} \rightarrow 1 : 1 \]

\[ \Rightarrow \delta \]

\[ [\star] \rightarrow \delta(\cdot,[\star]) \]

(\( \delta \) is representable functor)

Thus: if \( X \) is a G-set, the space \( |UoX| \)
has one connected component for each element of \( X \),
each connected component is contractible.

So we've "pulled up" \( X \), cofibrantly replacing by a big component.
\[ \text{Prf (of first claim) it suffices to show that two vertices of } \lbrack U_0, X \rbrack, \text{ i.e. elements of } G \times X, \text{ map to the same element of } X \text{ by } \varphi_0, \text{ iff they're connected by a 1-simplex } \rightarrow \text{ suppose } (g, x) \text{ and } (h, y) \text{ have } gx = hy, \text{ unit an edge in } \tilde{X} \text{ connecting them: } (g, x) \rightarrow (g_0, g_1, z) \rightarrow (h, y) \]

\[ \text{i.e. } (g_0, g_1, z) \in G \times G \times X \text{ such that } g_0 g_1 = h \text{ and } z = y, \]
\[ g_0 = g \text{ and } g_1 z = x \]
\[ \Rightarrow g_1 = g^{-1} h \text{ (the "proof" that } gx = hy) \]

Conversely, any two vertices connected by an edge map to same element of \( X \):

\( \text{(because } g_0 (g, x) = (g_0 g) x \text{ )} \)

will show contractible later.

This works for any monadic adjunction.

\[ \text{Q&A: what about monoid-sets? (secretly didn't need the inverse, explained directed, homotopy? (slow-going))} \]