10/29 Bar Construction for Groups

given any group G, we have an adjunction

$$\begin{tikzcd} \mathrm{Set} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathrm{Set}^G \arrow[l, bend left, "U"] \end{tikzcd}$$

given any G-set X, the counit $\varepsilon_X : FU(X) \to X$ sends $(g, x) \mapsto gx$. (this is the face map).

the bar construction gives a simplicial ~~G-set~~

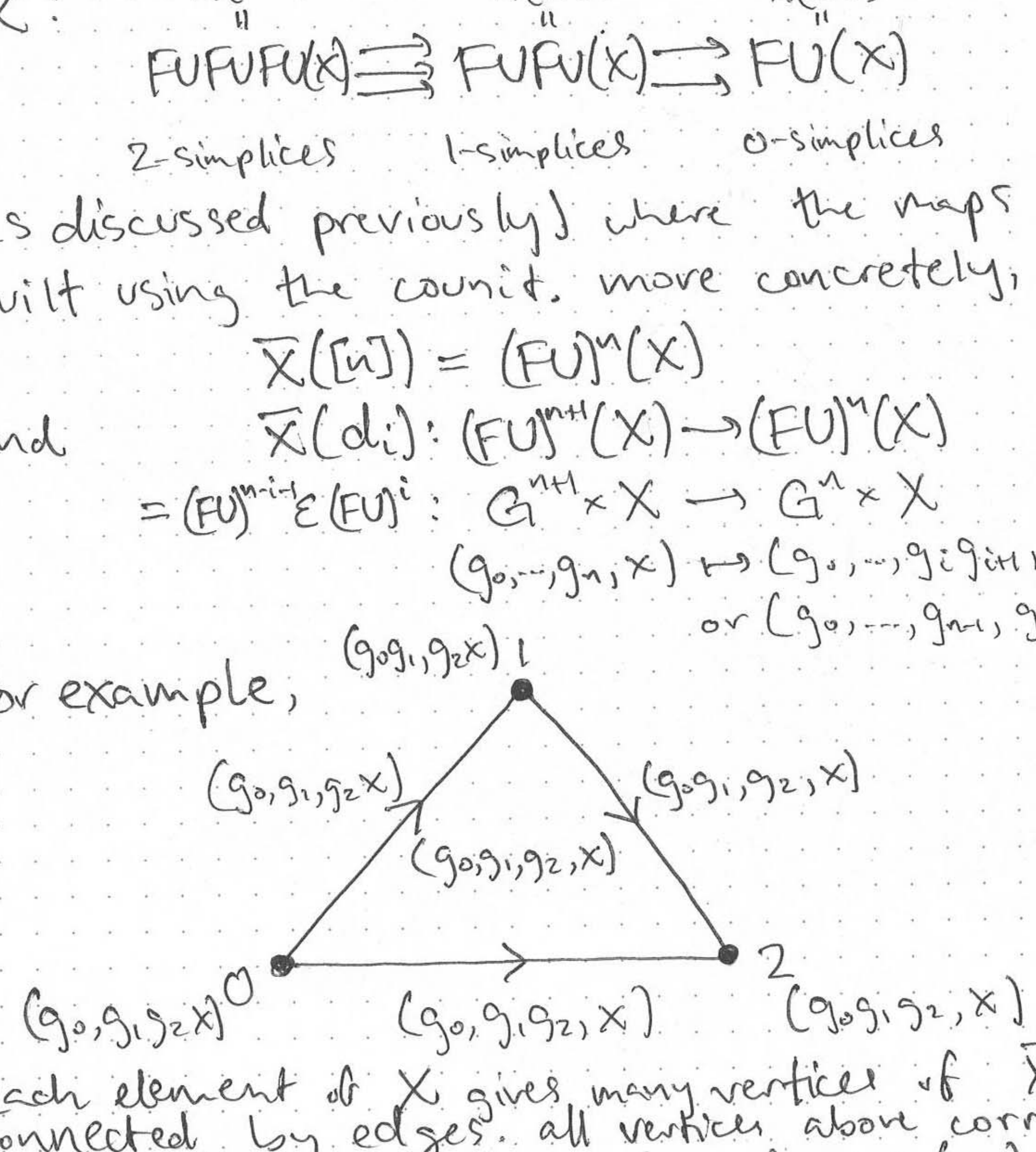
$$\overline{X}: \quad \begin{array}{ccccc} \overline{X}([3]) & & \overline{X}([2]) & & \overline{X}([1]) \\ \| & & \| & & \| \\ FUFUFU(X) & \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} & FUFU(X) & \rightrightarrows & FU(X) \\ & & & & \\ \text{2-simplices} & & \text{1-simplices} & & \text{0-simplices} \end{array}$$

(as discussed previously) where the maps are built using the counit. more concretely,

$$\overline{X}([n]) = (FU)^n(X)$$

and
$$\overline{X}(d_i) : (FU)^{n+1}(X) \to (FU)^n(X)$$
$$= (FU)^{n-i-1}\varepsilon(FU)^i : G^{n+1} \times X \to G^n \times X$$
$$(g_0, \dots, g_n, x) \mapsto (g_0, \dots, g_i g_{i+1}, \dots, g_n, x)$$
$$\text{or } (g_0, \dots, g_{n-1}, g_n x)$$

for example,

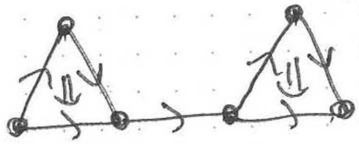


each element of X gives many vertices of $\overline{X}$, connected by edges. all vertices above correspond to same element of X: $g_0(g_1g_2x) = g_0g_1(g_2x) = (g_0g_1g_2)x$.

the bar construction realizes proofs
(the edges are the proofs of equality)
— this is a crucial idea of modern mathematics.

Our simplicial G-set $\overline{X}$ has an underlying
simplicial set $U(\overline{X}) : \Delta^{op} \to Set^{G} \to Set$.
Then we can turn any simplicial set into a
topological space:

Thm: there's a functor called geometric realization
$$|\cdot| : Set^{\Delta^{op}} \to Top$$
Prf: (sketch) — "visually evident"



To prove it, there's a functor
(category theorists could talk about
this diagram for days)

$$\begin{tikzcd} \Delta \arrow[r, hook] \arrow[d] & Set^{\Delta^{op}} \arrow[dl, dashed, "|\cdot|"] \\ Top & \end{tikzcd}$$

we can turn any simplex into a space;
this functor extends to geometric realization
using the Yoneda embedding, $\Delta \hookrightarrow Set^{\Delta^{op}}$
sends each simplex to simpset which looks like that simplex. $[n] \longmapsto \Delta(-,[n])$
(representable functor)

Thm: if X is a G-set, the space $|U \circ \overline{X}|$
has one connected component for each element of X.
each connected component is contractible.
So we've "puffed up" X, cofibrantly replacing each element by a big component.

Prf (of first claim) it suffices to show that two vertices of $|U \circ \overline{X}|$, ie elmts of $G \times X$, map to the same elemt of X by ε_X, iff they're connected by a 1-simplex

$\rightarrow$ suppose (g,x) & (h,y) have $gx = hy$, want an edge in $\overline{X}$ connecting them:

$$(g,x) \overset{(g_0, g_1, z)}{\longrightarrow} (h,y)$$

ie a $(g_0, g_1, z) \in G \times G \times X$ such that

$g_0 g_1 = h$ and $z = y$

$g_0 = g$ and $g_1 z = x$

$\Rightarrow g_1 = g^{-1} h$ (the "proof" that $gx = hy$)

conversely, any two vertices connected by an edge map to same elmt of X:

(because "$g_0 (g_1 x) = (g_0 g_1) x$")

will show contractible later ∎

This works for any monadic adjunction.

Q&A: what about monoid-sets?

[secretly didn't need the inverse, explain later]

directed homotopy?

(slow-going)