Group Cohomology

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recall: if X is a G-set, $\overline{X}$ is a simplicial G-set:

$$\cdots G\times G\times G\times X \rightrightarrows G\times G\times X \rightrightarrows G\times X$$

with face maps $\overline{X}(d_i): G^{n+1}\times X \to G^n \times X$

"multiplies across ith comma" $\quad g(g_0,\dots,g_{n-1},x)=(gg_0,\dots,x)$

(note that these $G^n\times X$ are still G-sets, with action)

now we'll finally do some cohomology.

if $X=1=\{*\}$, we call $\overline{X}$ "$\mathbb{E}G$", looks like:

$$G\times G\times G \rightrightarrows G\times G \rightrightarrows G$$

with $\mathbb{E}G(d_i)(g_0,\dots,g_n)=\begin{cases}(g_0,\dots,g_ig_{i+1},\dots,g_n) & 0\le i<n\\ (g_0,\dots,g_{n-1}) & i=n\end{cases}$

<u>dfn</u> the geometric realization $|\mathbb{E}G|$ $= EG \in Top$, the "<u>universal contractible G-space</u>"

(recall $|\cdot|: Set^{\Delta^{op}} \to Top$, but $\overline{(-)}: Set \to (Set^G)^{\Delta^{op}}$, so we're actually doing

$$(Set^G)^{\Delta^{op}} \cong [\Delta^{op}\times G, Set] \cong [G,[\Delta^{op}, Set]]$$
$$\downarrow |\cdot|$$
$$\text{"}G\text{-spaces"} \longrightarrow [G, Top]$$

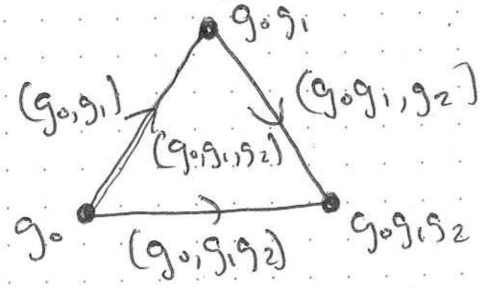
also last time claimed each component contractible)

since $X=1$, EG is contractible.

universal — some kind of initial property

(if Z is contractible G-space, $\exists!\, EG \xrightarrow{\varphi} Z$ in Top^G)

a typical 2-simplex in $\mathbb{E}G$:



(associativity)

note: • there's one vertex for each group element

• there's one edge from any vertex to any other

$$g \xrightarrow[(g,\, g^{-1}h)]{} h \quad \text{(self-action)}$$

• there's one triangle $\forall (g,h,k) \in G^3$, expressing associativity

• one n-simplex $\forall (\,,\,) \in G^n$ etc.

we can turn $\mathbb{E}G$ into a simplicial abelian group using the free functor $Set \to Ab$, which we'll call $\mathbb{Z}[-] :: X \mapsto \mathbb{Z}[X] \sim \mathbb{Z}$-linear combinations of elements of X. $\mathbb{Z}[\mathbb{E}G]$ looks like:

$$\cdots \mathbb{Z}[G\times G\times G] \overset{\partial_0,\ \partial_1,\ \partial_2}{\rightrightarrows} \mathbb{Z}[G\times G] \overset{\partial_0}{\underset{\partial_1}{\rightrightarrows}} \mathbb{Z}[G]$$

with maps as before on generators.

we can then turn it into a chain complex:

$$\cdots \longrightarrow \mathbb{Z}[G^3] \xrightarrow{\partial} \mathbb{Z}[G^2] \xrightarrow{\partial} \mathbb{Z}[G]$$

with $\partial : \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^n] = \sum_{i=0}^{n} (-1)^i \partial_i$

(note: $\mathbb{Z}[\mathbb{E}G]$ is also a simplicial abelian G-module, ie still has an action of G)

— on generators, $g(g_0, \ldots, g_n) = (gg_0, \ldots, g_n)$