

so our chain complex is one of G -modules
 and G -module maps ($\partial g(c) = g\partial(c)$)

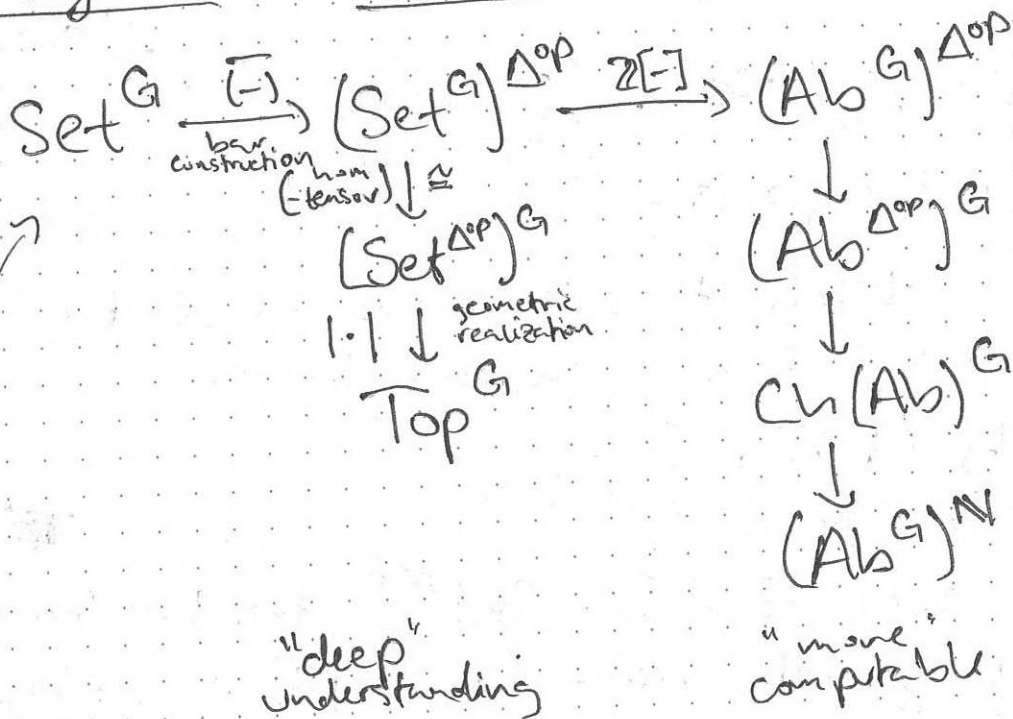
Group Cohomology (cont)

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Traditional Algebra

Homotopy Theory

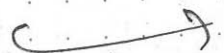
Homology Theory

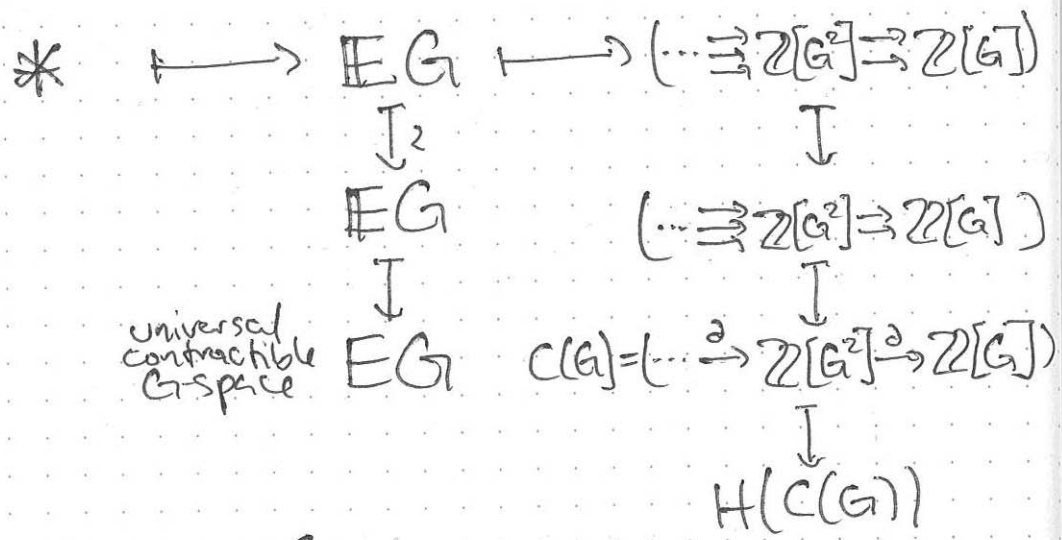


Can do this for many categories.
 another important one is

$R\text{Mod}$ — cohomology of rings

now let's apply this machine to $*$,
 the 1-element set is a G -set with
 trivial action:





Thm: $H_n(C(G)) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$
 — like the point!

Prf: (sketch) this is because we started with a point; EG is contractible, so

$$H_n(EG) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$$

singular (nth homology on top.sp.)

in general if $S \in \text{Set}^{\Delta^{op}}$, then the homology of its geometric realization $|S|$ is naturally isomorphic to the homology of $C(\mathbb{Z}[S])$
 — the chain complex of the simp. ab. $\mathbb{Z}[S]$.

we have a map of chain complexes

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & \mathbb{Z}[G^3] & \xrightarrow{d} & \mathbb{Z}[G^2] & \xrightarrow{d} & \mathbb{Z}[G] & \Sigma a_i g_i \\
 & & \downarrow & & \downarrow & & \downarrow & \downarrow \\
 \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}[G^0] & \Sigma a_i
 \end{array}$$

which induces an isomorphism on homology. (using the thm)

this ~~domain~~ domain complex is the "free resolution" of the ~~domain~~ codomain — the $\mathbb{Z}[G^n]$ are free- G modules with the same homology, a fancy puffed-up version of the bottom row.

Dfn: given any G -module A , let

$$C^n(G, A) := \text{Ab}^G(C_n(G), A)$$

and $d: C^n(G, A) \rightarrow C^{n+1}(G, A)$

be $df(c) = f(\partial c)$

where $c \in C^{n+1}(G)$, $\partial c \in C^n(G)$, and $f \in C^n(G, A)$ so $f(\partial c) \in A$.

Dfn: the group cohomology of a group G with "coefficients" in a G -module A is:

$$H^n(G, A) := \frac{\ker(d: C^n(G, A) \rightarrow C^{n+1}(G, A))}{\text{im}(d: C^{n+1}(G, A) \rightarrow C^n(G, A))}$$

(note - inker because $(d^2 f)(c) = (df)(\partial c) = f(\partial^2 c) = 0$.)

★ we'll see $H^2(G, A)$ classifies short exact sequences
 $0 \rightarrow A \rightarrow X \rightarrow G \rightarrow 0$ (this was the origin of cohomology)

— ie, ways of glomming A & G together to form a bigger group.

$H^3(G, A)$ classifies ways of glomming A & G together to form a "2-group": a category \mathcal{C} with $1 \xrightarrow{I} \mathcal{C} \xrightarrow{m} \mathcal{C}^2$ obeying group laws up to iso.

Q&A: how are we getting 2-groups??

| | | | |
|---------|----------|---|----------|
| | π_1 | } | G |
| | π_2 | | A |
| abelian | π_3 | | 0 |
| | \vdots | | \vdots |

←
this works for
topological spaces,
simplicial sets,
 ∞ -groupoids