so our chain complex is one of G-modules and G-module maps $(\partial g(c) = g\partial(c))$

Group Cohomology (cont) 11/2

Traditional Algebra	Homotopy Theory	Homology Theory

$$
\begin{tikzcd}
\mathrm{Set}^G \arrow[r, "{[-]}", "\text{bar construction}"'] & (\mathrm{Set}^G)^{\Delta^{op}} \arrow[r, "{\mathbb{Z}[-]}"] \arrow[d, "\cong", "\text{hom (-tensor)}"'] & (\mathrm{Ab}^G)^{\Delta^{op}} \arrow[d] \\
 & (\mathrm{Set}^{\Delta^{op}})^G \arrow[d, "|\cdot|"', "\text{geometric realization}"] & (\mathrm{Ab}^{\Delta^{op}})^G \arrow[d] \\
 & \mathrm{Top}^G & \mathrm{Ch}(\mathrm{Ab})^G \arrow[d] \\
 & & (\mathrm{Ab}^G)^{\mathbb{N}}
\end{tikzcd}
$$

"deep" understanding

"more" computable

Can do this for many categories.
another important one is

RMod — cohomology of rings

now let's apply this machine to $*$, the 1-element set as a G-set with trivial action:

$$\begin{tikzcd}
* \arrow[r, mapsto] & \mathbb{E}G \arrow[r, mapsto] \arrow[d, mapsto, "2"] & (\cdots \substack{\to\\\to\\\to} \mathbb{Z}[G^2] \rightrightarrows \mathbb{Z}[G]) \arrow[d, mapsto] \\
& \mathbb{E}G \arrow[d, mapsto] & (\cdots \substack{\to\\\to\\\to} \mathbb{Z}[G^2] \rightrightarrows \mathbb{Z}[G]) \arrow[d, mapsto] \\
& \text{universal contractible } G\text{-space} \quad EG & C(G) = (\cdots \xrightarrow{\partial} \mathbb{Z}[G^2] \xrightarrow{\partial} \mathbb{Z}[G]) \arrow[d, mapsto] \\
& & H(C(G))
\end{tikzcd}$$

<u>Thm</u>: $H_n(C(G)) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$

— like the point!

<u>Prf</u>: (sketch) this is because we started with a point; EG is contractible, so

$$\underset{\substack{\text{singular} \\ (\text{nth homology on top.sp.})}}{H_n}(EG) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n>0 \end{cases}$$

in general if $S \in \mathrm{Set}^{\Delta^{op}}$, then the homology of its geometric realization $|S|$ is naturally isomorphic to the homology of $C(\mathbb{Z}[S])$ — the chain complex of the simp. ab. $\mathbb{Z}[S]$.

"∎"

we have a map of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[G^3] \arrow[r, "\partial"] \arrow[d] & \mathbb{Z}[G^2] \arrow[r, "\partial"] \arrow[d] & \mathbb{Z}[G] \arrow[d] & \sum a_i g_i \arrow[d, mapsto] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}([G^0]) & \sum a_i
\end{tikzcd}$$

which induces an isomorphism on homology. (using the thm) this domain complex is the "free resolution" of the codomain — the $\mathbb{Z}[G^n]$ are free-G modules with the same homology, a fancy puffed-up version of the bottom row.

<u>Dfn</u>: given any G-module A, let

$$C^n(G,A) := \mathrm{Ab}^G(C_n(G), A)$$

and

$$d: C^n(G,A) \to C^{n+1}(G,A)$$

be

$$df(c) = f(\partial c)$$

where $c \in C^{n+1}(G)$, $\partial c \in C^n(G)$, and $f \in C^n(G,A)$ so $f(\partial c) \in A$.

<u>Dfn</u>: the <u>group cohomology</u> of a group G with "coefficients" in a G-module A is:

$$H^n(G,A) := \frac{\ker(d: C^n(G,A) \to C^{n+1}(G,A))}{\mathrm{im}(d: C^{n-1}(G,A) \to C^n(G,A))}$$

(note- $\mathrm{im} \subset \ker$ because $(d^2 f)(c) = (df)(\partial c) = f(\partial^2 c) = 0$.)

☆ we'll see $H^2(G,A)$ <u>classifies</u> short exact sequences

$$0 \to A \xrightarrow{i} X \xrightarrow{p} G \to 0$$ (this was the <u>origin</u> of cohomology)

— ie, ways of glomming A & G together to form a bigger group.

!!!

$H^3(G,A)$ classifies ways of glomming A & G together to form a "2-group": a category C with

$$\begin{tikzcd} 1 \arrow[r, "I"] & C \arrow[loop below, "inv"] & C^2 \arrow[l, "m"'] \end{tikzcd}$$

obeying group laws up to iso.

Q & A: how are we getting 2-groups??

$$\begin{array}{r|ll} & \pi_1 & G \\ & \pi_2 & A \\ \text{abelian} & \pi_3 & 0 \\ & \vdots & \vdots \end{array}$$

this works for topological spaces, simplicial sets, ∞-groupoids