Extensions of Groups

The fundamental Thm. of Arithmetic says any (nonzero) natural number is a product of primes, uniquely up to reordering.

—Something similar is true for finite groups, where the “atoms” are the simple groups, those w/o nontrivial normal subgroups.

For finite abelian groups, the simple ones are \( \mathbb{Z}_p \) (for \( p \) prime); but not every finite abelian group is a product of \( \mathbb{Z}_p \). For example, the 4-elm. abns are \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) & \( \mathbb{Z}_4 \), which is built from two \( \mathbb{Z}_2 \)'s as an “extension”; it has \( \mathbb{Z}_2 \) as a normal subgroup; the quotient of which is \( \mathbb{Z}_2 \).

—but nonabelian is harder.

Jordan-Hölder Thm: if \( G \) is a finite group, there is a composition series for \( G \) — subgroups \( 1 = H_0 \leq H_1 \leq H_2 \leq \cdots \leq H_n = G \) st. (1) \( H_i \trianglelefteq H_{i+1} \) \( (1 \leq i < n) \) (2) \( H_{i+1}/H_i \) is simple.

But, this series is not unique; but the resulting simple groups \( H_{i+1}/H_i \) are unique up to reordering.
dfn: a group $E$ is an extension of $G$ by $N$ if $N \triangleleft E$ and $E/N \leq G$; i.e. $\varnothing \rightarrow N \rightarrow E \rightarrow G \rightarrow \varnothing$ is exact.

So to completely classify finite groups, we "just" need to classify finite simple groups (proof: ~10,000 pages) and then understand extensions. We'll just study abelian extensions $\varnothing \rightarrow A \rightarrow E \rightarrow G \rightarrow \varnothing$ where $A$ is abelian. These are classified by $H^2(G, A)$. (*For fully general extensions, we'll need "nonabelian cohomology", which uses even more 2-category theory.*)

suppose $1 \rightarrow A \rightarrow E \xrightarrow{p} G \rightarrow 1$ is an abelian extension; we can think of $A$ as a subgroup of $E$, so write $i(a) = a$. We can choose a function $j: G \rightarrow E$:

$1 \rightarrow A \rightarrow E \xrightarrow{i} G \rightarrow 1$

such $p \circ j = 1_G$, not necessarily a homomorphism, nor unique—but exists:

since elements of $G$ are $eA$ for $e \in E$, we can always choose $j(g)$ to be any element of the coset that is $g$; then $p(j(g)) = g$. 

\[
\begin{array}{c|c|c|c}
0 & 1 & 2 \\
3 & 4 & 5 \\
6 & 7 & 8 \\
\end{array}
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Then each element $e \in E$ is of the form $j(g) \cdot a$ for some unique $g \in G$ and $a \in A$; $g$ says which coset $e$ is in, and $a$ says where $e$ is in the coset. So we get a bijection

$$A \times G \to E$$

$$(a, g) \mapsto j(g) a$$

(whoops; switch to $a, j(g)$ below)

How does multiplication work in $E$, in terms of these pairs?

$$j(g) a \cdot j(g') a'$$

$$a \cdot j(g) a' = j(g)^{-1} j(g) j(g')$$

call this $\frac{A_{\text{normal}}}{e \in A}$

$$a \cdot x(g)a' \cdot j(g)j(g')j(g_{g'})j(g_{g''})$$

this $eA$ too!

call it $c(g, g', g'')$

Thm: $\alpha$ is an action of $G$ on $A$, making $A$ into a $G$-module, namely

$$(*) \quad \alpha(g)(aa') = (\alpha(g)a)(\alpha(g)a')$$

$$\alpha(gg')(a) = \alpha(g)(\alpha(g')a)$$

birth of group of conom

c: $G \times G \to A$ obeys the 2-cocycle condition:

$$\alpha(g)c(g', g'') - c(gg', g'') + c(g, g'g'') - c(g, g'') = 0$$

writing $G$-operation additively, for appearance)

hint: involves associativity...
This is how group cohomology was discovered. The connection to simplices involves the tetrahedron. In $G$, the associative law:

\[
\begin{align*}
\alpha \beta \gamma &= \alpha \beta \gamma \\
\gamma \alpha \beta &= \gamma \alpha \beta \\
\gamma \beta \alpha &= \gamma \beta \alpha
\end{align*}
\]

in $E = A \times G$, we have:

\[(a, g)(a', g') = (a + \alpha(g) a' + \alpha(g) g', g g')\]

and associativity of this is connected to the 2-cocycle condition:

\[
\begin{align*}
\alpha \beta \gamma &= \alpha \beta \gamma \\
\gamma \alpha \beta &= \gamma \alpha \beta \\
\gamma \beta \alpha &= \gamma \beta \alpha
\end{align*}
\]