Group Extns (cont.) 11/7

last time we almost showed:

<u>Thm</u> given group $(G, \cdot, 1)$ and abelian $(A, +, 0)$ there's a short exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{p} G \to 1$$

where $E = A \times G$ with multiplication:

$$(a, g)(a', g') = (a + \alpha(g)a' + c(g, g'), gg')$$

with $\quad i(a) = (a, 1)$; $p(a, g) = g$,

and ① $\alpha : G \to \mathrm{Aut}(A)$, so A is a G-module

② $c : G \times G \to A$ is a <u>2-cocycle</u>:

$$\alpha(g)c(g', g'') - c(gg', g'') + c(g, g'g'') - c(g, g') = 0$$

(forgot last time) * ③ c is <u>normalized</u>, conditions coming from E has identity $(0,1)$ and inverses, including:

(degeneracy maps) (later)
$$c(1, g) = c(g, 1) = 0 \quad \forall g \in G$$

moreover, every extension of G by A is isomorphic to one of this form.

<u>Prf</u> in HW, took any extn and showed ①/② and converse, is just a direct check. ∎

There are four cases:

	$\alpha(g) = 1_A$	α nontrivial
$c(g, g') = 0$	direct product $A \times G$	semidirect product $A \rtimes G$
c nontrivial	central extension of G by A	extension of G by A

(need that $A \subset Z(E)$
$\alpha(g)a = j(g)\,a\,j(g)^{-1} = a$,
hence commute with everyone)

Consider 8-element groups; none is simple, so all are built up by extensions:

① $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ — direct product

$\mathbb{Z}_2 \times \mathbb{Z}_4$

② dihedral D_8 — semidirect $(\mathbb{Z}_4 \rtimes \mathbb{Z}_2)$

reflections of □ act on rotation

③ $\mathbb{Z}_8$ is a central extension of $\mathbb{Z}_4$ by $\mathbb{Z}_2$

(also of $\mathbb{Z}_2$ by $\mathbb{Z}_4$)

④ Q_8 quaternions

$= \langle i, j, k : i^2, j^2, k^2 = -1 ;\ ij = -ji = k$ & cyclic permutations$\rangle$

$$1 \longrightarrow \underset{\mathbb{Z}_2}{\underset{\parallel}{\{\pm 1\}}} \longrightarrow Q_8 \longrightarrow \underset{\{i,j,k\}}{\underset{\parallel}{\mathbb{Z}_2 \times \mathbb{Z}_2}} \longrightarrow 1$$

hence also a central extension

– but more exciting b/c nonabelian.

Of the ~50 billion groups of order <2000, over 99% have order $1024 = 2^{10}$.

"Most" finite groups have order a power of 2

– they're built from $\mathbb{Z}_2$ using repeated extns.

(2 is smallest prime, so highest exponent gives the most combinations)

– Baez has to leave early for jury duty...