11/9 a 2-cocycle in everyday life: an odometer described $\mathbb{Z}_{1000000}$ as an iterated central extension involving 6 copies of $\mathbb{Z}_{10}$. consider $\mathbb{Z}_{100}$ —

"carry the 1" is a cocycle

$$\begin{array}{r} \overset{+1}{1}9 \\ 23 \\ \hline 42 \end{array}$$

we write $\mathbb{Z}_{100}$ as $\mathbb{Z}_{10} \times \mathbb{Z}_{10}$ with addition:

$$(a,b)+(a',b') = (a+a'+c(b,b'), b+b')$$

where $c(b,b')$ is the carry digit.
(associativity gives $c$ the cocycle property)

<u>Group Cohomology</u>: using the bar construction, we turned the one-point G-set $*$ into a chain complex of G-modules:

$$\mathrm{Set}^G \xrightarrow{\overline{(-)}} (\mathrm{Set}^G)^{\Delta^{op}} \xrightarrow{\mathbb{Z}[-]} (\mathrm{Ab}^G)^{\Delta^{op}} \xrightarrow{\mathrm{ch}} \mathrm{Ch}(\mathrm{Ab}^G)$$

$$* \longmapsto \cdots \to \mathbb{Z}[G^2] \xrightarrow{\partial} \mathbb{Z}[G]$$

The cohomology of G with coefficients in the G-module A is cohomology of the cochain complex

$$\mathrm{Ab}^G(\mathbb{Z}[G^n], A)$$

with $d$ that increases $n$ by one. $(df(c) = f(\partial c))$
Let's simplify this! There's an isomorphism

$$\beta: \mathrm{Ab}^G(\mathbb{Z}[G^{n+1}], A) \xrightarrow{\sim} \mathrm{Ab}(\mathbb{Z}[G^n], A)$$

$$f \longmapsto \hat{f}$$

where $\hat{f}(g_1, \dots, g_n) = f(1, g_1, \dots, g_n)$
note: $f(g_0, \dots, g_n) = f(g_0(1, g_1, \dots, g_n))$

$= \alpha(g_0) f(1, g_1, \dots, g_n)$ (action $\alpha$: of $G$)

$= \alpha(g_0) \tilde{f}(g_1, \dots, g_n)$

(so we can recover $f$ from $\tilde{f}$, so $\beta$ is an isomorphism)

and now we can define $\tilde{\partial}$ uniquely, so that

$$\begin{array}{ccc} Ab^G(\mathbb{Z}[G^n], A) & \xrightarrow[\sim]{\beta} & Ab(\mathbb{Z}[G^{n-1}], A) \\ d \downarrow & \circlearrowright & \downarrow \tilde{\partial} \\ Ab^G(\mathbb{Z}[G^{n+1}], A) & \xrightarrow[\sim]{\beta} & Ab(\mathbb{Z}[G^n], A) \end{array}$$

$\tilde{\partial} = \beta \circ d \circ \beta^{-1}$, or $\tilde{\partial}\tilde{f} = \widetilde{df}$

now let's work out what this really is:

$$\tilde{\partial}\tilde{f}(g_1, \dots, g_n) = \widetilde{df}(g_1, \dots, g_n)$$
$$= df(1, g_1, \dots, g_n)$$
$$= f\, \partial(1, g_1, \dots, g_n)$$
$$= f(g_1, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i f(1, g_1, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^n f(1, g_1, \dots, g_{n-1})$$
$$= \alpha(g_1) \tilde{f}(g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i \tilde{f}(g_1, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^n \tilde{f}(g_1, \dots, g_{n-1}).$$

ex an extension of $G$ by $A$ gives a map $c: G^2 \to A$;
this is a 2-cocycle, i.e. $\tilde{\partial} c = 0$, iff

$$\alpha(g_1) c(g_2, g_3) - c(g_1 g_2, g_3) + c(g_1, g_2 g_3) - c(g_1, g_2) = 0$$

This is exactly the condition needed for our extension built using $\alpha$ + $c$ to be associative!

(just like the homework)

Thm: extensions of a group $G$ by an abelian group $A$ are classified, up to isomorphism, by

- an action $\alpha$ of $G$ by $A$
- a cohomology class $[c] \in H^2(G,A) = \dfrac{\{2\text{-cocycles}\}}{\{2\text{-coboundaries}\}}$

Idea of Proof starting from an extension

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

and a map $j: G \to E$ with $pj = 1_G$, we got an action $\alpha$ & a 2-cocycle $c$.
if we changed $j$, we'd get the same $\alpha$, and $c$ would change to $c' = c + \partial k$ for some $k: G \to A$.
conversely, given an action $\alpha$, and a 2-cocycle $c$, we can build a group structure on $E = A \times G$ and an extension $1 \to A \to E \to G \to 1$.
if we chose $c'$ with $[c] = [c']$, then $c' = c + \partial k$, we get another extension, but it's isomorphic:

$$\begin{tikzcd}
 & & E \arrow[dd, "\ell", "\wr"'] \arrow[dr] & & \\
1 \arrow[r] & A \arrow[ur] \arrow[dr] & & G \arrow[r] & 1 \\
 & & E' \arrow[ur] & &
\end{tikzcd}$$