11/16 Jury Duty: no more class except Fridays
(except day after Thanksgiving & Mon 11/26)
– but optionally extended for a 2nd hour.

examples of Group Cohomology

- suppose p, q prime.

<u>Prop</u> Let $\mathbb{Z}_q$ be made into a $\mathbb{Z}_p$-module in a trivial way. (identity automorphisms of $\mathbb{Z}_q$)
then
$$H^2(\mathbb{Z}_p, \mathbb{Z}_q) = \begin{cases} \mathbb{Z}_p & p=q \\ 0 & p \neq q \end{cases}$$
"$\mathbb{Z}_p$'s don't like to talk to each other"

$\Rightarrow$ the only central extension of $\mathbb{Z}_p$ by $\mathbb{Z}_q$ is trivial: $0 \to \mathbb{Z}_q \to \mathbb{Z}_q \oplus \mathbb{Z}_p \to \mathbb{Z}_p \to 0$
but when $p=q$, we get p nonisomorphic central extensions:
$$0 \to \mathbb{Z}_p \to E \to \mathbb{Z}_p \to 0$$
but in all of these $E \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$

(note $\mathbb{Z}_{pq} \cong \mathbb{Z}_p \oplus \mathbb{Z}_q$)

- much more generally, ① <u>Fracture Theorem</u>:
any group of order p^n is <u>nilpotent</u>:
built from abelian groups by central extensions
(<u>solvable</u> is built from abels by any extensions)

$\longrightarrow$

$$1 \to A_1 \to E_1 \to A_0 \to 1$$
$$1 \to A_2 \to E_2 \to E_1 \to 1$$
$$\vdots$$

(2) any group of order p^n is built from $\mathbb{Z}_p$ by iterated central extn (corollary)
— just show $\mathbb{Z}_{p^n}$ is a central extn of $\mathbb{Z}_{p^{n-1}}$ by $\mathbb{Z}_p$.

(nice, but complicated — remember 50B+ groups order ≤ 1000; full classification a long way off)

(3) any finite nilpotent group is a direct product of groups (necessarily nilpotent) of prime power order for various primes.

Higher-dimensional algebra (or "homotopification")

we can generalize familiar algebraic structures (groups, rings, ...) from sets to higher-dimensional things such as:
- categories (or 2-categories, ...)
- simplicial sets
- topological spaces

these choices are related

let Cat be the mere category of all (small) cats.
then we have functors

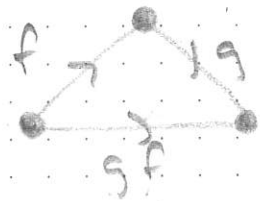
$$\mathrm{Cat} \xrightarrow[\text{"nerve"}]{N} \mathrm{Set}^{\Delta^{op}} \xrightarrow[\text{"realization"}]{|\cdot|} \mathrm{Top}$$

the nerve of a category C
has 0-simplices

$\bullet x$ objects of C

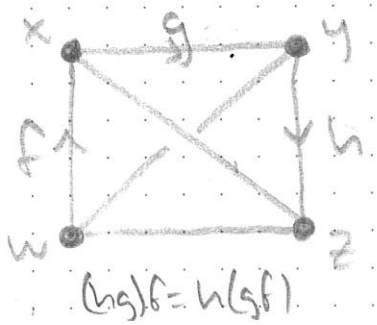
1-simplices $x \xrightarrow{f} y$ morphisms of C

2-simplices



commuting triangles in C

3-simplices



commuting tetrahedra in C

<u>dfn</u> given a category C, its <u>nerve</u>

$$N(C) : \Delta^{op} \longrightarrow \mathrm{Set}$$

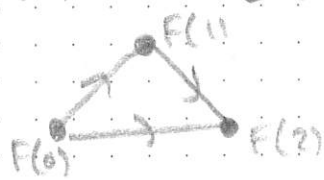
is given by $N(C)([n]) = \mathrm{Cat}([n], C)$

where $[n]$ is the totally ordered set $\{0, 1, \ldots, n-1\}$
which is a poset, hence a category:

$$0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-1$$

for example, $N(C)([3]) = \{\text{functors } F : [3] \to C\}$

$= \{$



$\}$

Thm ① every CW-complex is homotopy equivalent to $|X|$, for some $X \in \mathrm{Set}^{\Delta^{op}}$.

② every space $|X|$ is homeomorphic to $|N(C)|$ for some category C.

"category theory has a lot more topological meat than you may have thought".

Q&A:

$$\begin{tikzcd} \Delta \arrow[r, hook, "Y"] \arrow[d] & \mathrm{Set}^{\Delta^{op}} \arrow[dl, dashed, "|\cdot|"] \\ \mathrm{Top} & \end{tikzcd}$$

$|\cdot|$ preserves colimits

$\Rightarrow$ has a right adjoint

$$\mathrm{Top} \xrightarrow{\mathfrak{n}} \mathrm{Set}^{\Delta^{op}}$$

$$\mathfrak{n}(X)([n]) = \mathrm{Top}([n], X)$$

~ after hours ~

in algebraic topology we use this to turn a space into a chain complex:

$$\mathrm{Top} \xrightarrow{\mathfrak{n}} \mathrm{Set}^{\Delta^{op}} \xrightarrow{\mathbb{Z}[-]} \mathrm{Ab}^{\Delta^{op}} \xrightarrow{c} \mathrm{Ch}(\mathrm{Ab})$$

we have two adjunctions

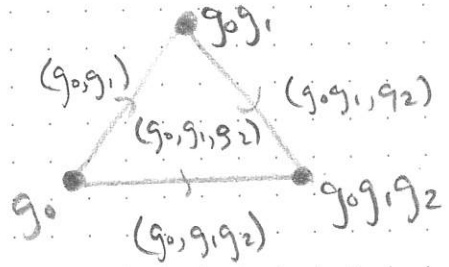
bar construction on these??

$$\begin{tikzcd} \mathrm{Cat} \arrow[r, bend left, "N"] \arrow[r, phantom, "\top"] & \mathrm{Set}^{\Delta^{op}} \arrow[l, bend left, "{[-]}"] \arrow[r, bend right, "|\cdot|"'] \arrow[r, phantom, "\top"] & \mathrm{Top} \arrow[l, bend right, "\mathfrak{n}"'] \end{tikzcd}$$

equally good for homotopy theory

(the theorem at the top is a big step toward proving that Cat, $\mathrm{Set}^{\Delta^{op}}$, and Top are "Quillen equivalent")

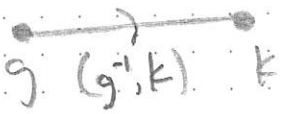
EG, revisited: given a group G, we get a simp. set EG



this is the <u>nerve</u> of some category!

- objects — group elements
- morphisms — exactly one morphism from any $g \in G$ to any $k \in G$

(so you know composition)



this is a (chaotic) groupoid, and an example:

<u>def</u> if X is a G-set with a right G-action, the <u>translation groupoid</u> $X//G$ has

(more gentle than quotienting)

- one object for each $x \in X$
- morphisms $x \xrightarrow{(x,g)} y$ (st $xg = y$)
- with obvious composition (like vectors)

<u>Thm</u> $N(G//G) = EG$

(Alex - similar to quiver representations)

the "weak quotient", rather than making things equal, gives specific isomorphisms between translated elements

there's a map $X//G \longrightarrow X/G$

that collapses isomorphisms to identities

(G/G is just a point, while

$|N(G//G)| = EG$ is a contractible space.

G acts trivially on a point, but

G acts nontrivially on EG, hence EG:

<u>Thm</u> let $BG = EG/G$ (already weak // right G, now strong / left G)

then: ① BG is connected,

and $\pi_1(BG) \cong G$,

$\pi_n(BG) \cong \{0\} \quad \forall n > 1$

② any CW complex with these properties is homotopy equivalent to BG.

math more connected than you thought

— group theorists study "BG"s