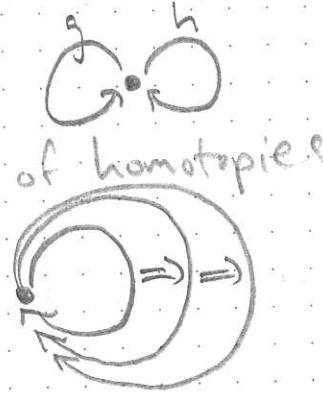


# $n$ -groupoids and the periodic table 12/7

using general (not strict) monoidal categories, we can define a notion of categorified group or "2-group":  $M$  is a monoidal category where ① every morphism is invertible (groupoid)  
② every object  $g \in M$  is invertible, ie  $\exists g^{-1} \in M$  such that  $g \otimes g^{-1} \cong I \cong g^{-1} \otimes g$ .

ex given a topological space  $X$  and a point  $x \in X$ , the fundamental 2-group of  $X$  has:

- objects are loops based at  $x$
- tensor is concatenation
- morphisms are (homotopy classes of) homotopies
- composition is that of homotopies



There's a concept of "equivalence" of monoidal categories (compatible with  $\otimes$ ), and thus of 2-groups. (might want to classify, but harder than groups.)

Thm every 2-group is (equivalent to) the fundamental 2-group of some space.  
(just like for 1-groups)

Thm any 2-group is equivalent to one built from these data



- a group  $G$  (elements = objects, multiplication =  $\otimes$ )
  - an abelian group  $A$   
(elements = endomorphisms  $a$  of unit object  $I$ )  
(multiplication = composition)
  - an action  $\rho: G \times A \rightarrow A$
  - a 3-cocycle  $c: G \times G \times G \rightarrow A$   
(coming from associator  
 $\alpha_{g,h,k}: (g \otimes h) \otimes k \xrightarrow{\sim} g \otimes (h \otimes k)$   
turned into a morphism  $c(g,h,k): I \rightarrow I$  <sup>in some clever way</sup>)
- Prf: later, but the 3-cocycle condition  
is equivalent to the pentagon identity for  $\alpha$ )

The story goes on: you can build an  $n$ -group from  $G, A, \rho$  & an  $(n+1)$ -cocycle  $c: G^{n+1} \rightarrow A$ .

What's an  $n$ -group? Let's see...

we saw that a "strict" monoidal category

is a 2-category with one object.

a general "weak" monoidal category

is a bicategory with one object.

just like a 2-category, except composition  
of 1-morphisms is associative + unital only  
up to natural isomorphisms;

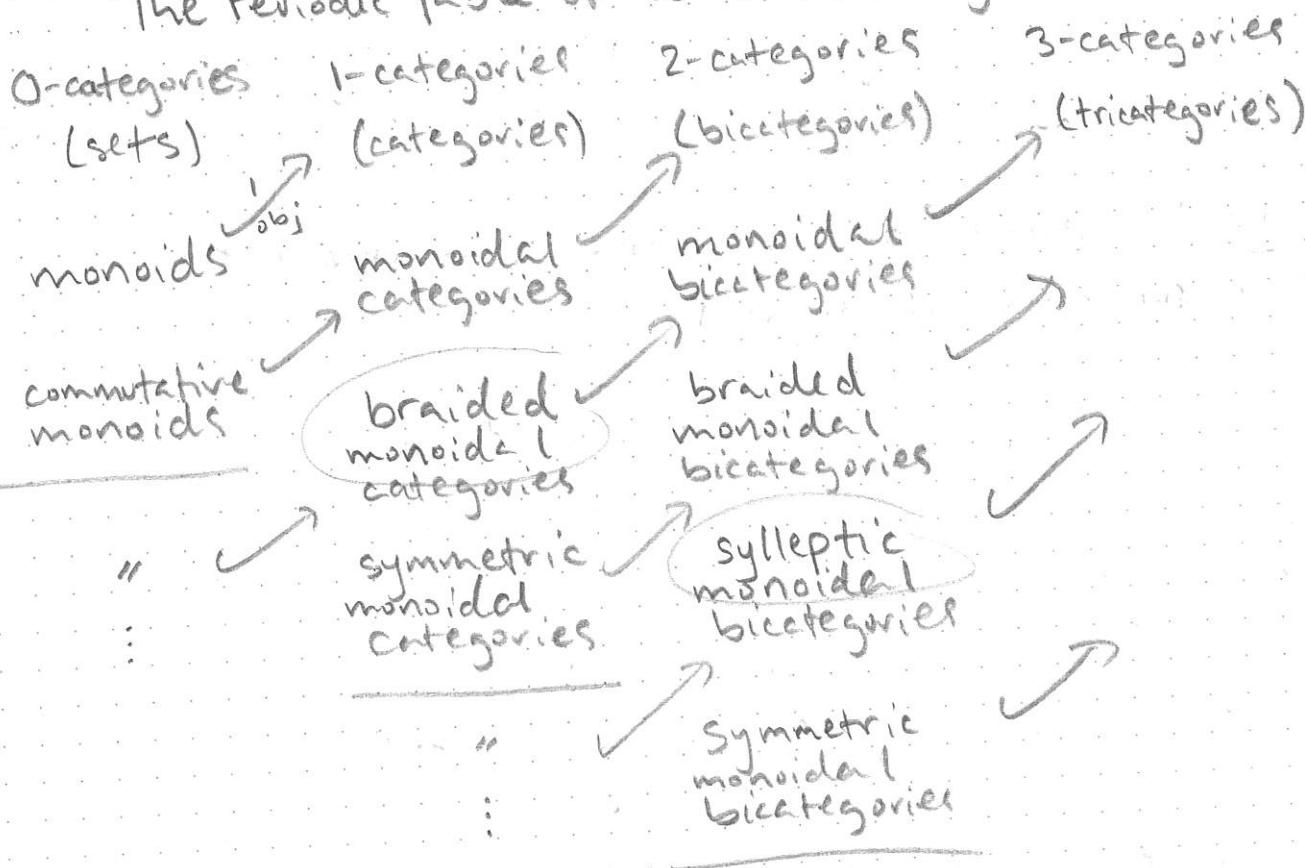
$(f \circ g) \circ h \cong f \circ (g \circ h)$

$\lambda_g: 1 \circ f \cong f$

$\rho_g: f \circ 1 \cong f$

obeying the pentagon + triangle identities.  
in general, weak  $n$ -categories are more  
challenging than strict ones.

### The Periodic Table of Weak $n$ -Categories



the stabilization hypothesis says that the  $n$ -category column "stabilizes" after  $n+2$  rows.  
(this has now been proved!)

a weak  $n$ -category where all  $j$ -morphisms ( $1 \leq j \leq n$ ) are invertible (up to a higher morphism) is called a weak  $n$ -groupoid. any space  $X$  has a fundamental  $n$ -groupoid  $\text{Th}_n(X)$ :

- objects are points
- morphisms are paths
- 2-morphisms are homotopies
- $n$ -morphisms are homotopy classes of

The Homotopy Hypothesis implies that every  $n$ -groupoid is the  $\text{Th}_n(X)$  of some space  $X$ .

(so Grothendieck understood homotopy theory is the study of these)

— the generalization to  $n$ -categories gives a directedness which is not present in conventional topology.  
fin.

### epilogue:

why do 2-groups all come (up to equivalence) from quadruples  $(G, A, \rho, c)$ ?

Thm: any 2-group is equivalent to one where:

- ① the left/right unitors are identities
- ② it's skeletal: isomorphic objects are equal

but we can still have a nontrivial associator

— but by (2) we have  $(g \otimes h) \otimes k = g \otimes (h \otimes k)$  ★  
 (Mac Lane's strictification theorem for 2-cats  
 has to throw in tons of objs, so not skeletal)

Let's consider a 2-group  $M$  of this form  
 & get  $(G, A, \circ, \otimes)$  from it.

- let  $G$  be the set of objects, with operation  $\otimes$   
 (by ★, really is associative, and same for inverses)
- let  $A = M(I, I)$   
 (clearly a group — why abelian?)

Eckmann-Hilton: note that

$$\circ : M(I, I) \times M(I, I) \rightarrow M(I, I)$$

(and)  $\otimes : M(I, I) \times M(I, I) \rightarrow M(I \otimes I, I \otimes I) = M(I, I)$

and the functoriality of  $\otimes$  also implies

$$(a \circ b) \otimes (c \circ d) = (a \otimes c) \circ (b \otimes d)$$

"the interchange law"

and  $1_I$  is the identity for both  $\otimes$  and  $\circ$  in  $A$ .

$$a \circ b = (a \otimes 1_I) \circ (1_I \otimes b)$$

$$= (a \circ 1_I) \otimes (1_I \circ b)$$

$$= a \otimes b$$

$$= (1_I \circ a) \otimes (b \circ 1_I)$$

$$= (1_I \otimes b) \circ (a \otimes 1_I) = b \circ a$$

secretly, commutativity is arising from the 2-dimensionality in a bicategory

(a 2-group is a one-object 2-groupoid)

so:  $A$  is an abelian group.

next, what's the action  $\rho: G \times A \rightarrow A$ ?

$$\rho(g) a = 1_g \otimes a \otimes 1_{g^{-1}}: g \otimes I \otimes g^{-1} \rightarrow g \otimes I \otimes g^{-1}$$

(but  $g \otimes I \otimes g^{-1} = I$ , so  $\rho(g)a \in A$ )

(you can check that it's an action)

what about all the other morphisms of  $M$ ?

skeletality: since they're isos, they're autos!

$f: g \xrightarrow{\sim} g$  given any of these,  
how to make  $I \rightarrow I$ ?

take  $f \otimes 1_{g^{-1}}$

i.e.,  $f = a \otimes 1_g$ , for some  $a \in A$ .

hence morphisms in  $M$  are pairs  $(a, g) \in A \times G$   
(be thinking of semidirect product!)

so finally, what about  $c$ ?

given  $g, h, k \in G^3$  we get  $\alpha_{g,h,k}: g \otimes h \otimes k \rightarrow g \otimes h \otimes k$

this corresponds to a pair  $(c(g, h, k), g \otimes h \otimes k)$

$\in A \times G$

the pentagon identity must be equivalent to some equation involving  $c$ :

$$\begin{array}{ccc}
 (g \otimes (h \otimes k)) \otimes l & \xrightarrow{\alpha_{g,h,k,l}} & g \otimes ((h \otimes k) \otimes l) \\
 \downarrow \alpha_{g,h,k \otimes l} & & \downarrow \alpha_{g,h,k,l} \\
 ((g \otimes h) \otimes k) \otimes l & & \\
 \downarrow \alpha_{g,h,k,l} & & \\
 (g \otimes h) \otimes (k \otimes l) & \xrightarrow{\alpha_{g,h,k \otimes l}} & g \otimes (h \otimes (k \otimes l))
 \end{array}$$

"action"

in terms of  $c$ , this says: (writing + for A)

$$\begin{aligned}
 & c(g, h, k) + c(g, h \otimes k, l) + p(g)c(h, k, l) \\
 & = c(g \otimes h, k, l) + c(g, h, k \otimes l)
 \end{aligned}$$

This is the 3-cocycle condition in group cohomology!

above is equivalent to:

$$\begin{aligned}
 & p(g)c(h, k, l) - c(gh, k, l) + c(gh, k, l) \\
 & - c(g, h, kl) + c(g, h, k) = 0 !!
 \end{aligned}$$

arrived at this from very different source)

- group cohomology, by bar construction
- pentagon, by higher categories

Let's continue systematising this beautiful stuff.