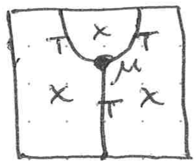
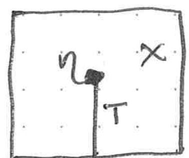
- adjunctions
- monads & comonads
- simplicial sets 
- cohomology
- classify group extensions $0 \to A \to B \to C \to 0$

Monads

defn: if X is a 2-category, a monad in X consists of

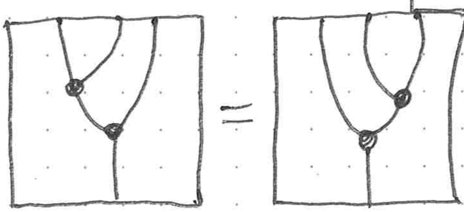
1. an object $x \in X$
2. a morphism $T : x \to x$
3. two 2-morphisms
   - multiplication $\mu : T \circ T \Rightarrow T$
   - unit $\eta : 1_x \Rightarrow T$



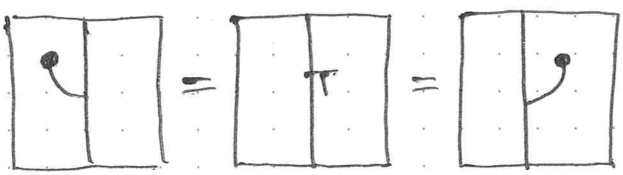


4. with properties
   - associativity

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"'] & T \end{tikzcd}$$



   - unitality



$$\begin{tikzcd} 1_x \circ T \arrow[d, Rightarrow, "\eta \circ T"'] & = & T \arrow[d, Rightarrow, "1_T"] & = & T \circ 1_x \arrow[d, Rightarrow, "T \circ \eta"] \\ T \circ T \arrow[rr, Rightarrow, "\mu"'] & & T & & T \circ T \arrow[ll, Rightarrow, "\mu"] \end{tikzcd}$$

Thm: if X is a 2-category containing an adjunction

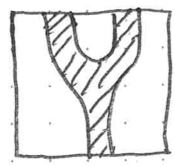
$$\begin{tikzcd} C \arrow[r, bend left, "F"] & D \arrow[l, bend left, "U"] \end{tikzcd}$$

then we get a monad,

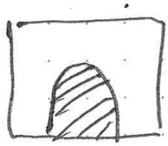
① C as the object

② $T = U \circ F$

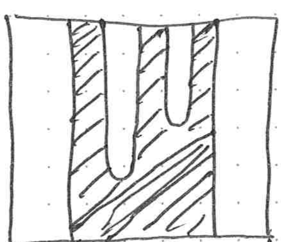
③ $\mu = U \varepsilon F : UFUF \Rightarrow UF$



$\eta = \eta$



Proof:



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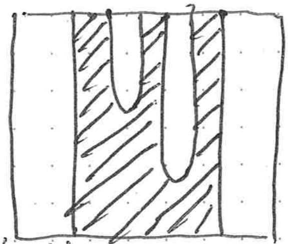
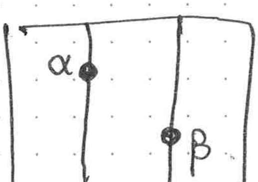


by, associativity

in general (



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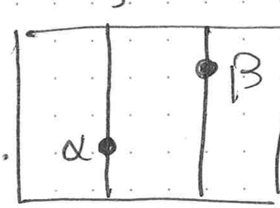
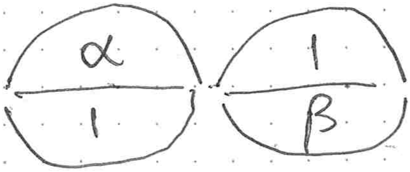


)

interchange law



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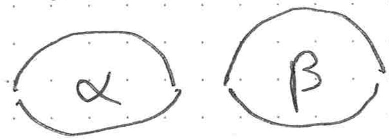

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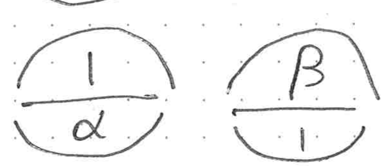

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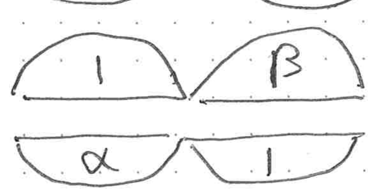
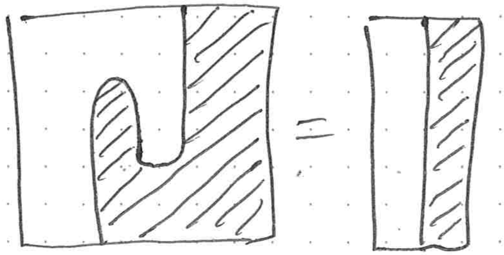
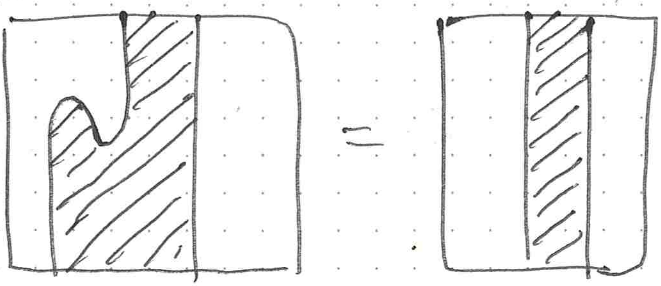


and left unit law by

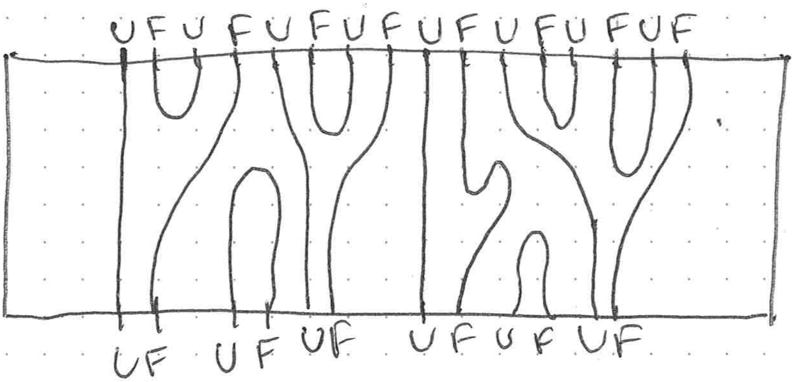


implies

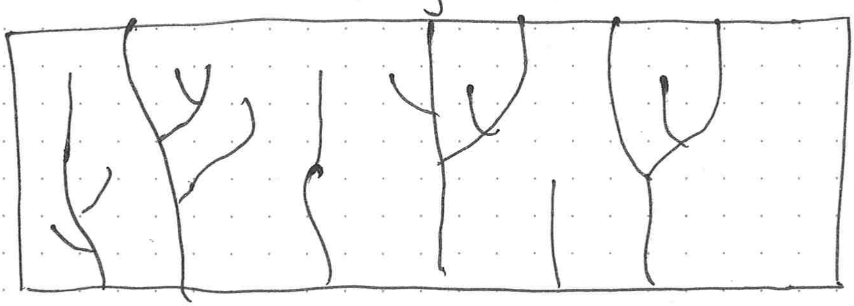


∎

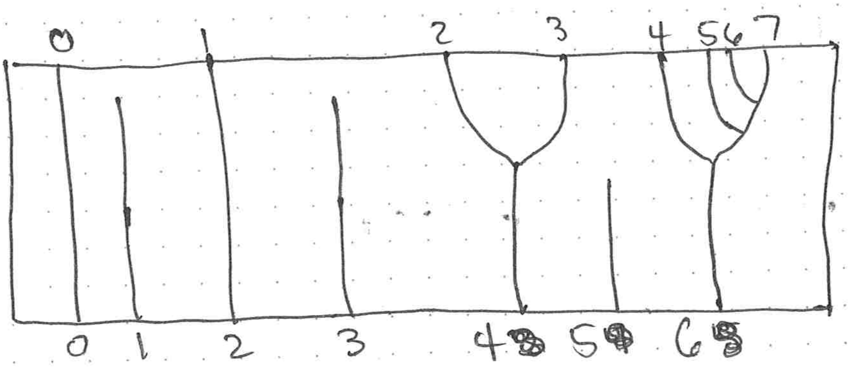
given an adjunction, we've seen that a general morphism in the category hom(C,C) looks like:



or more simply:

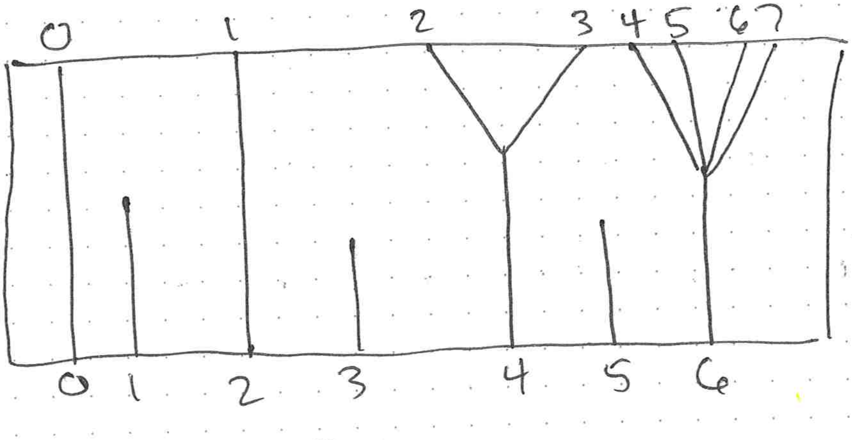


then the sadistic tree-trimmers come to town:



(normal form shortens the branchless 1,3,5)

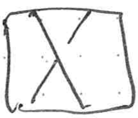
use notation which implies associativity:



what is this?

an <u>order-preserving</u> function of <u>finite linear orders</u>

we're seeing a model of Δ_a in hom(C,C)

(no crossings )

defn the <u>augmented simplex category</u>
Δ_a has:
- finite ordinals as objects

$$[0] = \{\ \}$$
$$[1] = \{0\}$$
$$[2] = \{0,1\}$$

- order-preserving maps as morphisms

Δ, the simplex category,
is the full subcategory of Δ_a
containing only nonempty ordinals
$$[1], [2], [3], \ldots$$
and all order-preserving maps

<u>Q&A</u>: - a "copy" of Δ_a means a functor
$$\Delta_a \to \hom(C,C)$$
- (there is "the walking adjunction")

$$\forall \text{adj} \in X \quad \exists!\ 2\text{-functor } \mathrm{Adj} \to X$$

- interesting adj's besides Cat?
(think of monoidal