**The Walking Monad**

defn: given 2-categories $C$ & $D$, a 2-functor $F: C \to D$ consists of:

- a function (called $F$) sending objects of $C$ to objects of $D$

- given $c, c' \in C$, a functor (called $F$)

$$F: C(c, c') \to D(F(c), F(c'))$$

such that these preserve composition & identity

defn: the **walking monad** $M$ is the 2-category with:

- one object $* \in M$

- $M(*, *) = \Delta_0$,

the augmented simplex category

(with objects finite ordinals, morphisms order-preserving maps)

comp. functor $M(*, *) \times M(*, *) \to M(*, *)$

looks like these:

$\Box \alpha: [0+1] \to [1+1]$  
$([m+n] \to [n+m])$
only possible identity:

Note: composition in \( M(\ast, \ast) \)
is "vertical composition"

\[ \alpha \cdot \beta = \]

(cone-object 2-category:
\( \approx \) strict monoidal category, "\( \cdot \)"

Thm if \( C \) is any 2-category, there is
a 1-1 correspondence between monads in \( C \)
and 2-functors \( M \rightarrow C \)

Prf let \( F: M \rightarrow C \) be a 2-functor:
want a monad \( x \in C, T: x \Rightarrow x, \mu: T \Rightarrow T, \text{unit/assoc} \)

let \( x = F(\ast) \)
\( T = F([1]) \)
\( \mu = F(\boxtimes) \)
\( i = F(\square) \)
\( \mu \circ i \) obey assoc/l\text{r} units
because they hold in \( M \),
and \( F \) preserves them.

Converse: basically same backwards
the simplex category, $\Delta$

recall, has nonempty finite ordinals and order-preserving functions; its objects can be drawn as simplices:

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow \\
\end{array}$$

its morphisms are certain maps between simplices:

$$\begin{array}{ccc}
& & [2] \\
& \rightarrow & \\
[1] & \rightarrow & [3]
\end{array}$$

the map $d_i : [n] \rightarrow [n+1]$ is the unique order-preserving injection with $i$ not in its range ($0 \leq i \leq n$) (there are also other maps.)

a simplicial set is a functor

$F : \Delta^\text{op} \rightarrow \text{Set}$

so for each $n \in \mathbb{N}$, $F([n])$ is the “set of $n$-simplices”. each order-preserving map $g : [m] \rightarrow [n]$ gives a function $F(g)$:

$F(g) : F([m]) \rightarrow F([n])$ contravariant
For example, $\partial_i : [n] \to [n+1]$ will give function $F(\partial_i) : F([n+1]) \to F([n])$ "boundary maps" picking out the $i$th face of each simplex in $F([n+1])$.

So a simplicial set could look like this:

$$F = \begin{array}{c}
\text{A} \\
\text{B} \\
\text{2} \\
1
\end{array}$$

$$A \in F([3])$$

$$B = F(\partial_1)(A) \in F([2])$$

(Like higher-dimensional tinker toys)

—a CW complex built up from $n$-balls (simplices equivalent, but connecting rules seem more rigid; but there's a highly nontrivial theorem that every CW is homotopy equivalent to a simplicial set.)

So, we can build up essentially all spaces we need in a "simplistic" way.

Q&A: Yoneda: $F([3]) \cong \left[ \begin{array}{c} \text{\triangle} \\ F \end{array} \right]$

boundary? ($\Sigma E_1$; etc.)—comes from (class, no risk)

Set free $A_k$

category of simplicial abelian groups is equivalent to category of chain complexes