10/15 The Walking Monad

defn: given 2-categories $\mathcal{C}$ & $\mathcal{D}$, a <u>2-functor</u> $F : \mathcal{C} \to \mathcal{D}$ consists of:

- a function (called F) sending objects of $\mathcal{C}$ to objects of $\mathcal{D}$.
- given $c, c' \in \mathcal{C}$ a functor (called F)

$$F : \mathcal{C}(c, c') \longrightarrow \mathcal{D}(F(c), F(c'))$$

$$\begin{tikzcd} c \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & c' \arrow[from=U, to=D, phantom, "\Downarrow \alpha"] \end{tikzcd} \longmapsto \begin{tikzcd} F(c) \arrow[r, bend left, "F(f)"{name=U}] \arrow[r, bend right, "F(g)"'{name=D}] & F(c') \arrow[from=U, to=D, phantom, "\Downarrow F(\alpha)"] \end{tikzcd}$$

such that these preserve composition & identity.

defn: the <u>walking monad</u> $\mathbb{M}$ is the 2-category with

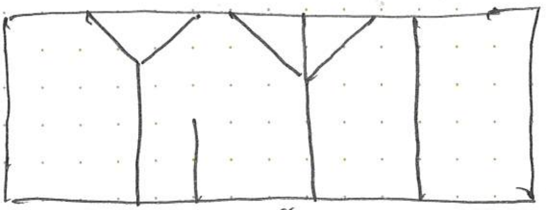
- one object $* \in \mathbb{M}$
- $\mathbb{M}(*, *) = \Delta_a$, the <u>augmented simplex category</u>

(with objects finite ordinals, morphisms order-preserving maps)

comp. functor • $\mathbb{M}(*, *) \times \mathbb{M}(*, *) \longrightarrow \mathbb{M}(*, *)$

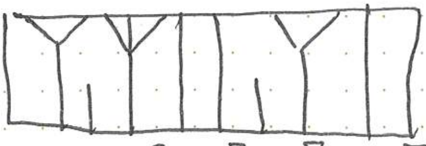
looks like these:

$[6]$



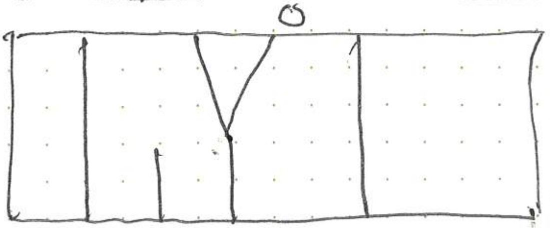
$= \beta$

$[4]$



$=$

$\circ$

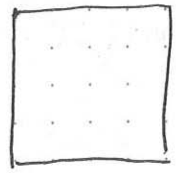


$= \alpha$

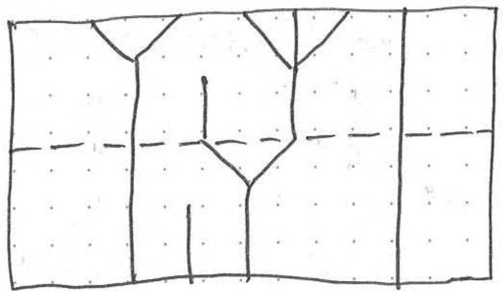
$[4]$

$\beta \circ \alpha : [6+4] \to [4+4]$

$([m+m'] \to [n+n'])$

• only possible identity: 

Note: composition in $M(*,*)$ is "vertical composition"

$\alpha \bullet \beta =$  [6] [4]

(one-object 2-category: $\approx$ strict monoidal category, "+")

<u>Thm</u> if C is any 2-category, there is a 1-1 correspondence between monads in C and 2-functors $M \to C$

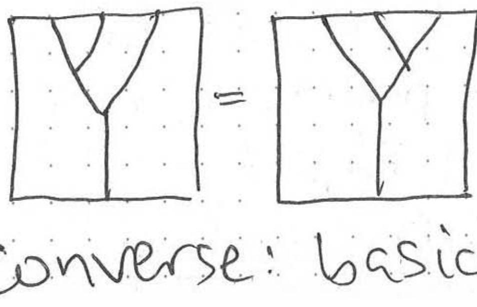
<u>Prf</u> let $F: M \to C$ be a 2-functor.
(want a monad $x \in C$, $T: x \to x$, $\mu: T^2 \Rightarrow T$, $i: 1_x \Rightarrow T$ unit/assoc)

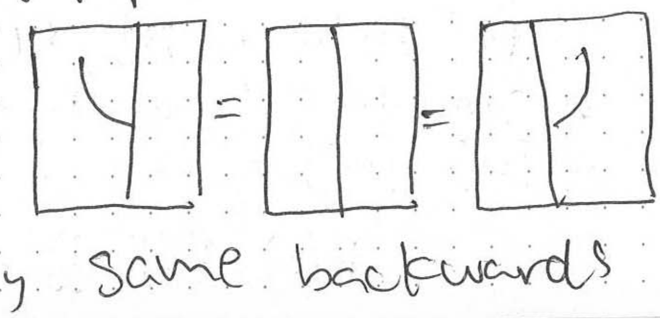
let $x = F(*)$

$T = F([1])$

$\mu = F(\boxed{Y})$ — $\mu + i$ obey assoc/ lr units because they hold in M, and F preserves them.

$i = F(\boxed{\mid})$



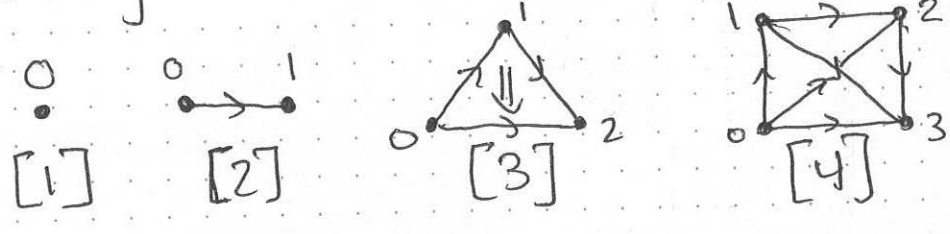


Converse: basically same backwards

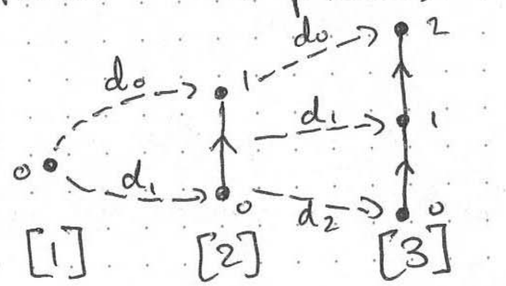
the simplex category Δ

recall, has <u>nonempty</u> finite ordinals
and order-preserving functions

its objects can be drawn as simplices



its morphisms are certain maps
between simplices:



---- (hard to draw)

the map $d_i : [n] \to [n+1]$
is the unique order-preserving injection
with i not in its range $(0 \leq i \leq n)$
(there are also other maps)

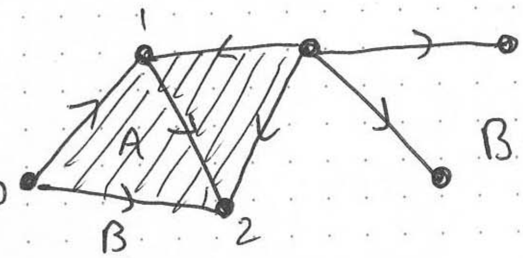
a <u>simplicial set</u> is a functor
$$F : \Delta^{op} \to Set$$
so for each $n \in \mathbb{N}$, $F([n])$ is the "set of
n-simplices". each order-preserving map
$g : [n] \to [m]$ gives a function ~~F(g): F(~~
$$F(g) : F([m]) \to F([n])$$ <u>contravariant</u>

for example, $d_i : [n] \to [n+1]$ will give function

$F(d_i) : F([n+1]) \to F([n])$ "boundary maps"

picking out the ith <u>face</u> of each simplex in $F([n+1])$.

So a simplicial set could look like this:

$F =$ 

$A \in F([3])$

$B = F(d_1)(A) \in F([2])$

(like higher-dimensional tinker toys)

— a CW complex built up from n-<u>balls</u> (simplices equivalent, but connecting rules seem more rigid; but there's a highly nontrivial theorem that every CW is homotopy equivalent to a simplicial set)

so, we can build up essentially all spaces we need in a "simplistic" way.

<u>Q&A</u>: Yoneda: $F([3]) \approx [\blacktriangle, F]$

boundary? ($\sum (-1)^i$ etc) — comes from (alg. top. class in a rush)

$$\mathrm{Set} \xrightarrow{\text{free}} \mathrm{Ab}$$

category of simplicial abelian groups is <u>equivalent</u> to category of chain complexes