Homology

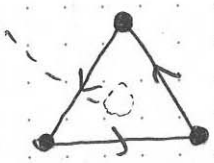
10/17

the plan now:

simplicial sets $\xrightarrow{\text{free}}$ simplicial abelian groups $\downarrow \wr$

abelian groups $\xleftarrow[\text{nth homology group}]{H_n}$ chain complexes of abelian groups

if a simplicial set has "holes of dimension n" they will be "detected" by H_n: group is nontrivial



has $H_1 \cong \mathbb{Z}$

*recall: a simplicial set is a functor

$$X: \Delta^{op} \to \text{Set}$$

thus a map of simplicial sets should be a natural transformation

$$\alpha: X \Rightarrow X'$$

α_n: $X([n])$ is set of n-simplices $\downarrow$ $X'([n])$ is set of n-simplices

such that α is "natural".

important to think about "simplicial ____" in lots of categories: given a category C, a simplicial object in C is a functor

$$X: \Delta^{op} \to C$$

(and maps between them give category $[\Delta^{op}, C]$)

- can you imagine anything more boring than sets? bring higher dimensions to everything! ("homotopification of mathematics")

for categories C, D, we write

(called exponent) $D^C := \hom_{Cat}(C, D)$ obj: functors, mor: nat trans

(meant to put C on bottom for consistency)

- so we're interested in $C^{\Delta^{op}}$ (simplicial objects in C)

given a simplicial set, we can turn it into a simplicial abelian group by:

$$\Delta^{op} \xrightarrow{X} Set \xrightarrow{free} Ab$$

(this generates formal linear combinations)

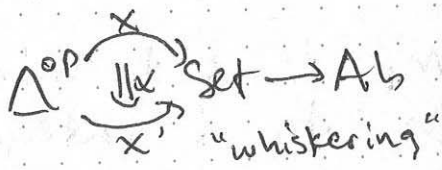
so $(f \circ X)([n])$ is the abelian group of of formal $\mathbb{Z}$-linear combinations of elements of $X([n])$. (like magic – introduces linear algebra)

in fact, this is a whole functor

$$Set^{\Delta^{op}} \longrightarrow Ab^{\Delta^{op}}$$

on objects: $X \mapsto f \circ X$

on morphisms: $\alpha \mapsto id_f \circ \alpha$



"whiskering"

- next, given a simplicial abelian group X, form a chain complex

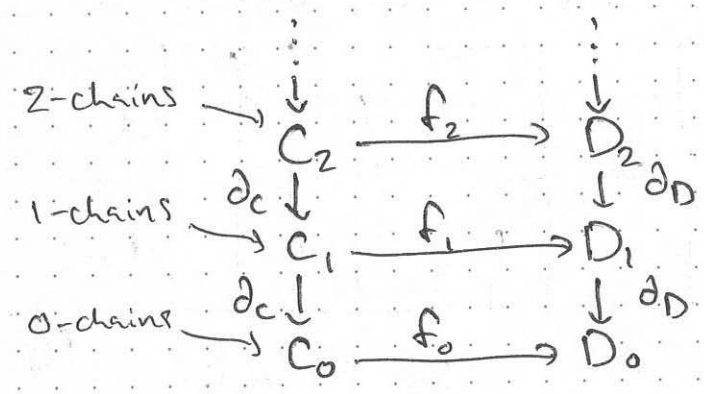
⟶

defn: a <u>chain complex</u> of abelian groups, $C_\bullet$ is a sequence of abgps + homomorphisms

$$\cdots \to C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

such that $\partial \circ \partial = 0$.

a <u>map</u> of chain complexes, $\phi: C \to D$, is



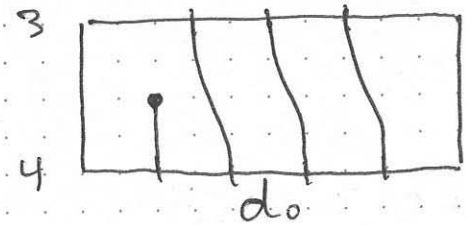
such that $f_i \circ \partial_C = \partial_D \circ f_{i+1}$

(forms category $Ch(Ab)$)

<u>Thm</u>: given $X \in Ab^{\Delta^{op}}$, we can form a chain complex $C(X) \in Ch(Ab)$ as follows:

$$C(X)_n = X([n])$$

and ∂ is defined as follows: in Δ, recall we have morphisms $d_i: [n+1] \leftarrow [n]$ $0 \le i \le n$



(four maps $[3] \to [4]$ which are 1-1 and order preserving)

so in Δ^{op} we have $d_i: [n+1] \to [n]$ and we get homomorphisms

$$\partial_i = X(d_i): C(X)_{n+1} \to C(X)_n$$

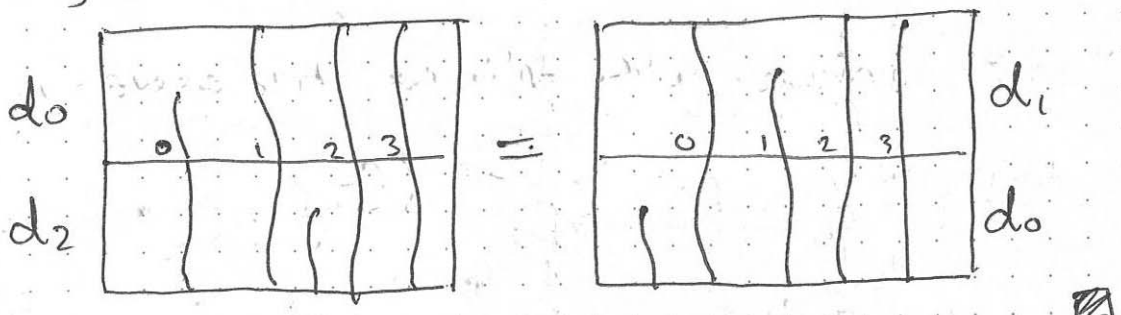
and $\partial : C(X)_{n+1} \to C(X)_n$ is defined by

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i \qquad \text{(but why?)}$$

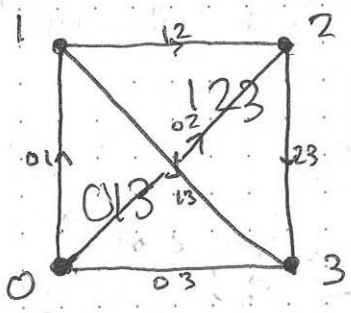
Prf the thing to prove is $\partial \circ \partial = 0$.

$$\partial \circ \partial = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} \partial_i \partial_j$$ which is zero because the terms cancel in pairs

using some relations that ∂_i obey:



→ example: the simplicial set (not $[3]$ itself) called "the walking 3-simplex"



this simplicial set contains a 3-simplex 0123 & so its chain complex has a 3-chain 0123

$$\partial(0123) = \sum_{i=0}^{3} (-1)^i \partial_i (0123)$$

$$= 123 - 023 + 013 - 012$$

$$\Rightarrow \partial^2(0123) = (23 - 13 + 12) - (23 - 03 + 02) + (13 - 03 + 01) - (12 - 02 + 01)$$

$$= 0.$$

(Eilenberg + MacLane used $\mathbb{Z}_2$ to avoid minus signs)