Homology

the plan now:

simplicial sets \xrightarrow{\text{free}} \text{simplicial abelian groups}
\downarrow
abelian groups \xrightarrow{H_n} \text{chain complexes of abelian groups}

if a simplicial set has "holes of dimension } n \)" they will be "detected" by } H_n : \text{group is nontrivial}

\begin{tikzpicture}
\node (A) at (0,0) {$*$};
\node (B) at (1,1) {$\bullet$};
\node (C) at (-1,1) {$\bullet$};
\draw (A) -- (B) -- (C) -- cycle;
\end{tikzpicture}

has } H_1 = 2

recall: a simplicial set is a functor $X : \Delta^{\text{op}} \rightarrow \text{Set}$

thus a map of simplicial sets should be a natural transformation $\alpha : X \Rightarrow X'$

$X([n])$ is set of $n$-simplices $\alpha_n : X([n]) \rightarrow X'([n])$ is set of $n$-simplices such that $\alpha$ is "natural",

important to think about "simplicial ---"

in lots of categories: given a category $C$ a simplicial object in $C$ is a functor $X : \Delta^{\text{op}} \rightarrow C$

(and maps between them give category $[\Delta^{\text{op}}, C]$)
can you imagine anything more boring than sets? bring higher dimensions to everything! ("homotopification of mathematics")

For categories $C, D$, we write
(called) $D^C := \hom_{\text{cat}}(C, D)$  \hspace{1cm} \text{obj: functors}  \hspace{1cm} \text{mor: nat trans}

(meant to put $C$ on bottom for consistency)

so we're interested in $C^{\Delta^0}$ (simplicial in $C$)
given a simplicial set, we can turn it into a simplicial abelian group by:

$$\Delta^0 \xrightarrow{X} \text{Set} \xrightarrow{\text{free}} \text{Ab}$$

(this generates formal linear combinations)

so $(f \circ X)([n])$ is the abelian group of formal $\mathbb{Z}$-linear combinations of elements of $X([n])$. (like magic - introduces linear algebra)

in fact, this is a whole functor

$$\text{Set}^{\Delta^0} \longrightarrow \text{Ab}^{\Delta^0}$$
on objects: $X \mapsto f \circ X$
on morphisms: $\alpha \mapsto \text{id}_{\Delta^0} \circ \alpha$

Next, given a simplicial abelian group $X$, form a chain complex
Defn: a chain complex of abelian groups, $C$, is a sequence of abgps $\{C_i\}$ and homomorphisms
$$\cdots \rightarrow C_3 \xrightarrow{d_3} C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$
such that $d_2 \circ d_1 = 0$.

A map of chain complexes, $\phi: C \rightarrow D$, is
$$\begin{array}{cccccc}
\text{2-chains} & i & C_2 & f_2 & D_2 & d_D \\
\text{1-chains} & d_c & C_1 & f_1 & D_1 & d_D \\
\text{0-chains} & d_0 & C_0 & f_0 & D_0 & \\
\end{array}$$
such that $f_i \circ d_c = d_D \circ f_i$, (forms category $\text{Ch}(\text{Ab})$)

Thm: given $X \in \text{Ab}$, we can form a chain complex $C(X) \in \text{Ch}(\text{Ab})$ as follows:
$C(X)_n = X([n])$

and $d$ is defined as follows:

\begin{align*}
\begin{array}{ccccccc}
\text{0} & \cdots & \text{3} & \text{4}
\end{array}
\end{align*}

\begin{array}{ccccccc}
\text{3} & \text{4} & \text{5} & \text{6} & \text{7}
\end{array}
\end{align*}

\begin{array}{ccccccc}
\text{3} & \text{4} & \text{5} & \text{6} & \text{7}
\end{array}
\end{align*}

\begin{array}{ccccccc}
\text{3} & \text{4} & \text{5} & \text{6} & \text{7}
\end{array}
\end{align*}

(3 maps $[3] \rightarrow [4]$, which are 1-1 and order preserving)

So in $\Delta^{op}$, we have $d_i: [n+1] \rightarrow [n]$ and we get homomorphisms
$\phi_i = X(d_i): C(X)_{n+1} \rightarrow C(X)_n$
and $d : C(X)_{n+1} \to C(X)_n$ is defined by

$$d = \sum_{i=0}^{n} (-1)^i d_i \quad \text{(but why?)}$$

**Proof.** The thing to prove is $d \circ d = 0$.

$$d \circ d = \sum_{i=0}^{n-1} \sum_{j=0}^{n} (-1)^{i+j} d_i \circ d_j$$

which is zero because the terms cancel in pairs using some relations that $d_i$ obey.

Example: the simplicial set called "the walking 3-simplex".

This simplicial set contains a 3-simplex 0123 and so its chain complex has a 3-chain $0123$.

$$d(0123) = \sum_{i=0}^{3} (-1)^i d_i(0123)$$

$$= 123 - 023 + 013 - 012$$

$$= 0 \quad \text{(Eilenberg & MacLane used } \mathbb{Z}_2 \text{ to avoid minus signs)}$$