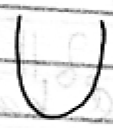
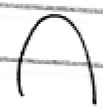
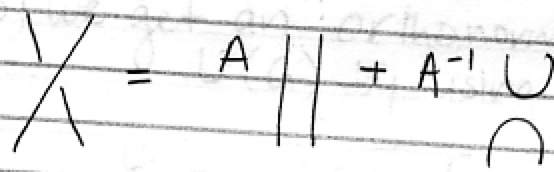
4/10/01

HW #1: want  $V \otimes V \to \mathbb{C}$ $\quad V = \mathbb{C}^2 \ni e_1, e_2$

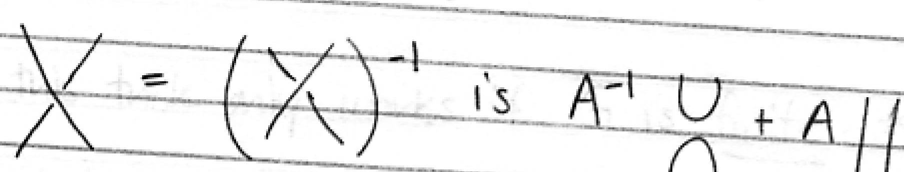
and  $\mathbb{C} \to V \otimes V$

st

and s.t. if



Then the inverse



soln: (when A=1)

$e_1 \otimes e_1$	$e_1 \otimes e_2$	$e_2 \otimes e_1$	$e_2 \otimes e_2$
$\mapsto$	$\mapsto$	$\mapsto$	$\mapsto$
0	1	-1	0

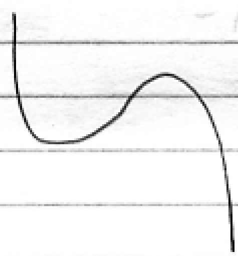
So guess: for other A:

$$\begin{array}{cccc} e_1 \otimes e_1 & e_1 \otimes e_2 & e_2 \otimes e_1 & e_2 \otimes e_2 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 0 & A & -A^{-1} & 0 \end{array}$$

Then:

$$\begin{array}{cc} \mathbb{C} & 1 \\ \downarrow & \downarrow \\ V \otimes V & e_1 \otimes e_2 - e_2 \otimes e_1 \end{array} \quad (\text{when } A = 1)$$

guess: $1 \longmapsto -Ae_1 \otimes e_2 + A^{-1} e_2 \otimes e_1$.



$$\begin{array}{cc} V & e_1 \\ \downarrow & \downarrow \\ V \otimes V \otimes V & -Ae_1 \otimes \underbrace{e_1 \otimes}_{=0} e_2 + A^{-1} \underbrace{e_1 \otimes e_2}_{=A} \otimes e_1 \\ \downarrow & \downarrow \\ V & e_1 \end{array}$$

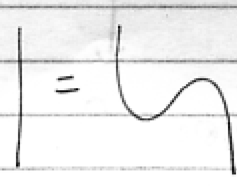
apply α to $\sqcup$

$A^{-1} A \otimes e_1 = e_1$

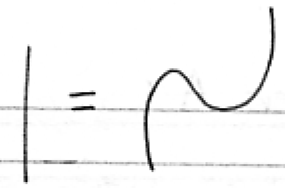
Now check what happens when we send in e_2.

$$e_2$$
$$\downarrow$$
$$-A \underbrace{e_2 \otimes e_1}_{} \otimes e_2 + A^{-1} \underbrace{e_2 \otimes e_2}_{=0} \otimes e_1$$
$$-A(-A^{-1}) \otimes e_2$$
$$\downarrow$$
$$e_2 \quad \checkmark$$

so



$\checkmark$

Now check 

e_1

$\downarrow$

$-Ae_1 \otimes \underbrace{e_2 \otimes e_1} + A^{-1} e_2 \otimes \underbrace{e_1 \otimes e_1}$

$-Ae_1 \otimes (-A^{-1}) + 0$

$\downarrow$

e_1 ✓

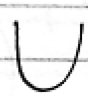
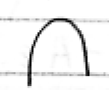
e_2

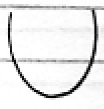
$\downarrow$

$-Ae_1 \otimes \underbrace{e_2 \otimes e_2}_{= 0} + A^{-1} e_2 \otimes \underbrace{e_1 \otimes e_2}_{A}$

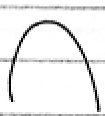
$\downarrow$

e_2 ✓

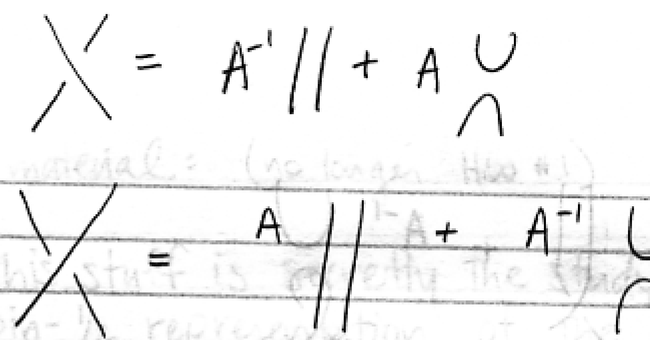
Now see what  does to 



$e_1 \otimes e_1$	$e_1 \otimes e_2$	$e_2 \otimes e_1$	$e_2 \otimes e_2$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	A	$-A^{-1}$	0



$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	$-A^2 e_1 \otimes e_2 + e_2 \otimes e_1$	$e_1 \otimes e_2 - A^{-2} e_2 \otimes e_1$	0



What does this do to the basis vectors:

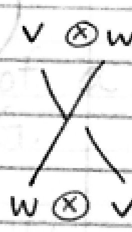
$e_1 \otimes e_1 \longrightarrow A e_1 \otimes e_1$

$e_1 \otimes e_2 \longrightarrow A e_1 \otimes e_2 - A e_1 \otimes e_2 + A^{-1} e_2 \otimes e_1 = A^{-1} e_2 \otimes e_1$

$e_2 \otimes e_1 \longrightarrow A e_2 \otimes e_1 + A^{-1} e_1 \otimes e_2 - A^{-3} e_2 \otimes e_1$

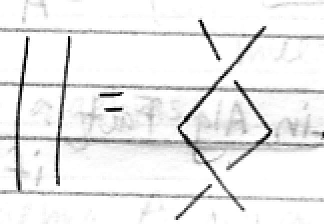
$e_2 \otimes e_2 \longrightarrow A e_2 \otimes e_2$

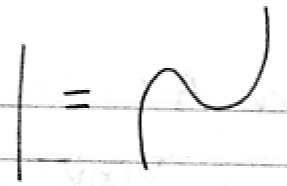
<u>Note</u>: when $(A=1)$ reduces to correctly switching 2 vectors



ex) $e_1 \otimes e_2 \longmapsto e_2 \otimes e_1$

We could just check that



Now check 

e_1

$\downarrow$

$-Ae_1 \otimes \underbrace{e_2 \otimes e_1} + A^{-1} e_2 \otimes \underbrace{e_1 \otimes e_1}$

$-Ae_1 \otimes (-A^{-1}) + 0$

$\downarrow$

e_1 ✓

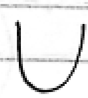
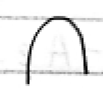
e_2

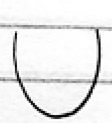
$\downarrow$

$-Ae_1 \otimes \underbrace{e_2 \otimes e_2}_{0} + A^{-1} e_2 \otimes \underbrace{e_1 \otimes e_2}_{A}$

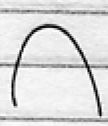
$\downarrow$

e_2 ✓

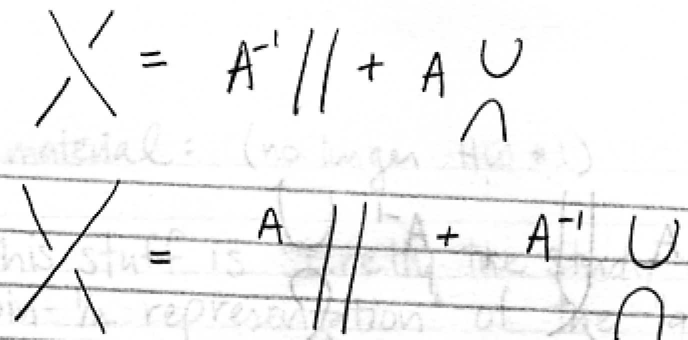
Now see what  does to 



$e_1 \otimes e_1$	$e_1 \otimes e_2$	$e_2 \otimes e_1$	$e_2 \otimes e_2$
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	A	$-A^{-1}$	0



$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	$-A^2 e_1 \otimes e_2 + e_2 \otimes e_1$	$e_1 \otimes e_2 - A^{-2} e_2 \otimes e_1$	0



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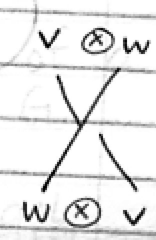
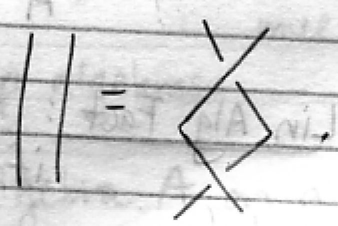
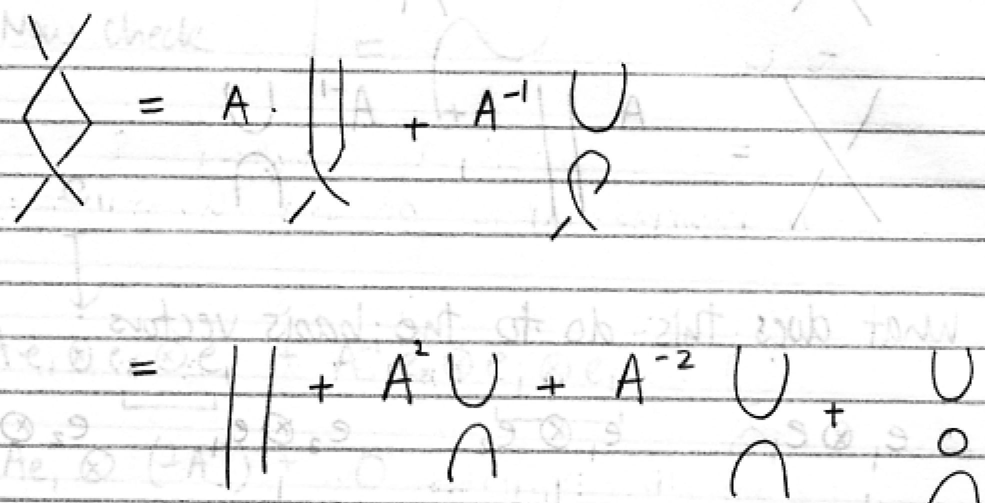


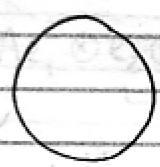
ex) $e_1 \otimes e_2 \longmapsto e_2 \otimes e_1$

We could just check that 



The last 3 will cancel if:

$$\bigcirc = -(A^2 + A^{-2})$$



$$\downarrow I$$

$$-Ae_1 \otimes e_2 + A^{-1} e_2 \otimes e_1$$

$$\downarrow \text{cup}$$

$$-A^2 - A^{-2} \quad \checkmark.$$

<u>Lin Alg Fact</u>: $ST = 1$ iff $TS = 1$
if
$S, T : L \longrightarrow L$ are linear
and L is finite dim'l.

New material: (no longer HW #1)

All this stuff is secretly the study of the "spin-½ representation" of the "quantum group" called

"quantum $SL(2, \mathbb{C})$"

and when $A = 1$, this reduces to the group $SL(2, \mathbb{C})$ - all transformations preserving the symplectic structure on $\mathbb{C}^2$.

Note: A quantum group is not a group!

It's an algebra (v.space w/ assoc. bilinear product & unit) that pretends to be a group.

(Last time - we showed how a group pretends to be an algebra!)

An example is the group algebra $\mathbb{C}[G]$ of a group: finite linear comb. of elts. of G. w/

$$\left(\sum a_i g_i\right)\left(\sum b_j h_j\right) = \sum a_i b_j \underbrace{g_i h_j}_{\text{mult. in } G.}$$

Quantum groups are like group algebras.

Defn: A representation of an algebra A on a v. space V is a linear map

$$\rho : A \longrightarrow \mathrm{End}(V) \quad \text{st}$$

$$\rho(aa') = \rho(a)\rho(a') \text{ and } \rho(1) = 1_V.$$

*we can turn our group into an alg. in this way.

Prop: A representation of the group G is the same thing as a rep. of the algebra $\mathbb{C}[G]$.
(ie- the two are in 1-1 correspondence).

proof: Given a rep. ρ of G, let $\tilde{\rho}$ be a rep. of $\mathbb{C}[G]$ by:

$$\tilde{\rho}(\Sigma a_i g_i) = \Sigma a_i \rho(g_i)$$

Given a representation ρ of $\mathbb{C}[G]$, let $\hat{\rho}$ be a rep. of G by:

$$\hat{\rho}(g) = \rho(g). \qquad G \subseteq \mathbb{C}[G].$$

$\square$

Given two reps. of G, say

$$\rho: G \longrightarrow \mathrm{End}(V)$$
$$\rho': G \longrightarrow \mathrm{End}(V')$$ we get a rep.

$$\rho \otimes \rho' : G \longrightarrow \mathrm{End}(V \otimes V')$$
by

$$(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$$

We'll check that this $\rho \otimes \rho'$ is a rep.
We can't do this trick to tensor reps. of algebras:

$$(\rho \otimes \rho')(a) = \rho(a) \otimes \rho'(a).$$

because: Need $\rho \otimes \rho' : A \longrightarrow \mathrm{End}(V)$ linear, but it's NOT!

proof that $(\rho \otimes \rho')$ isn't linear:

$$(\rho \otimes \rho')(2a) = \rho(2a) \otimes \rho'(2a)$$

$$= 4\rho(a) \otimes \rho'(a)$$

if $\rho \otimes \rho'$ were linear,

$$(\rho \otimes \rho')(2a) = 2\rho(a) \otimes \rho'(a).$$

We should be able to tensor reps. of group algebras, however.

In fact - we CAN tensor reps of a group algebra from what we've already said about group algs.

Given $\rho : \mathbb{C}[G] \longrightarrow \mathrm{End}(V)$

$\rho' : \mathbb{C}[G] \longrightarrow \mathrm{End}(V')$ we get:

$(\rho \otimes \rho')(g) = \rho(g) \otimes \rho'(g)$ $g \in$ group G.

so

$$(\rho \otimes \rho')\left(\sum a_i g_i\right) = \sum a_i\, \rho(g) \otimes \rho'(g)$$

since $(\rho \otimes \rho')$ as a rep. of an alg. must be linear.

Trick — a group alg. has as basis the elts. of the group.

This "duplication" of group elements gives us a map:

"duplication" map:

$$\Delta : \mathbb{C}[G] \longrightarrow \mathbb{C}[G] \otimes \mathbb{C}[G]$$

$$g \longmapsto g \otimes g$$

where $g \in G$ (the group) and g is a basis elt. for $\mathbb{C}[G]$. (elts of the group form a basis of the group alg).

w/ $\rho \otimes \rho' : \mathbb{C}[G] \longrightarrow \mathrm{End}(V \otimes V')$ given by:

$$\begin{array}{c} \mathbb{C}[G] \xrightarrow{\ \Delta\ } \mathbb{C}[G] \otimes \mathbb{C}[G] \\ \qquad\qquad\quad \Big\downarrow \rho \otimes \rho' \\ \qquad\qquad\quad \mathrm{End}(V) \otimes \mathrm{End}(V') \\ \qquad\qquad\quad \cong \\ \qquad\qquad\quad \mathrm{End}(V \otimes V') \end{array}$$

$$g \xmapsto{\ \Delta\ } g \otimes g \xmapsto{\ \rho \otimes \rho'\ } \rho(g) \otimes \rho'(g)$$

Suppose A is an algebra, $\Delta : A \longrightarrow A \otimes A$ is linear and

$$\rho : A \longrightarrow \mathrm{End}(V), \quad \rho' : A \longrightarrow \mathrm{End}(V').$$

We can try to define a rep. $\rho \otimes \rho' : A \longrightarrow \mathrm{End}(V \otimes V')$, by the same formula:

$$(*) \quad A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho \otimes \rho'} \mathrm{End}(V) \otimes \mathrm{End}(V') \cong \mathrm{End}(V \otimes V').$$

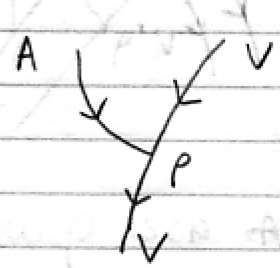
<u>BUT</u>: Is it a rep.?

Is $(\rho \otimes \rho')(aa') = (\rho \otimes \rho')(a) \cdot (\rho \otimes \rho')(a')$ and

$$(\rho \otimes \rho')(1) = 1_{V \otimes V'} \; ??$$

Let's use pictures to represent representations:

A rep. $\rho : A \longrightarrow \mathrm{End}(V)$ looks like:

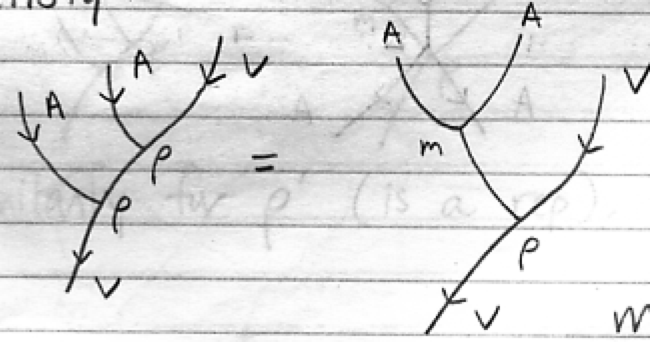


$$A \otimes V \xrightarrow{\tilde{\rho}} V$$

gives $\rho : A \longrightarrow \mathrm{End}(V)$

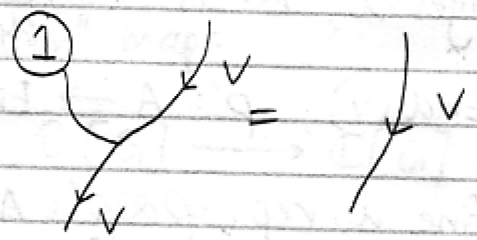
$$\rho(a)(v) = \tilde{\rho}(a \otimes v)$$

Must satisfy:



where m is mult.

$m : A \otimes A \longrightarrow A$ is mult. in A.

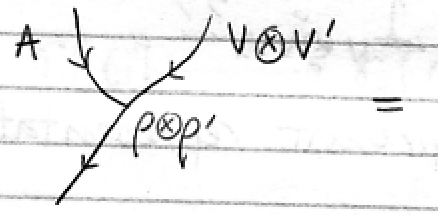


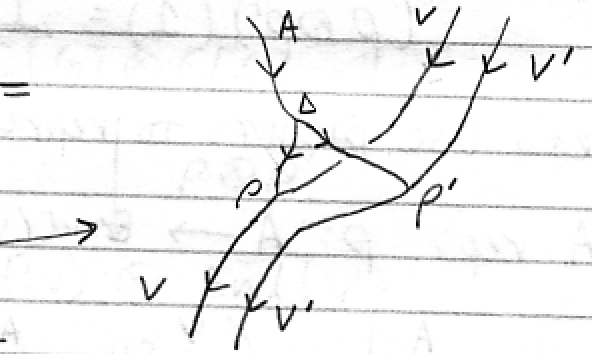
Given reps $\rho : A \longrightarrow \mathrm{End}(V)$

$\rho' : A \longrightarrow \mathrm{End}(V')$

what does $(\rho \otimes \rho') : A \longrightarrow \mathrm{End}(V \otimes V')$ look like?

we draw // lines for tensor product

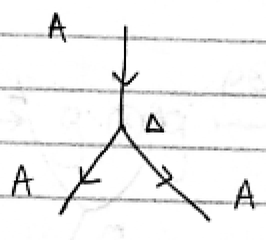




see (*) prev pg

we duplicate then break up.

We draw $\Delta : A \longrightarrow A \otimes A$ as



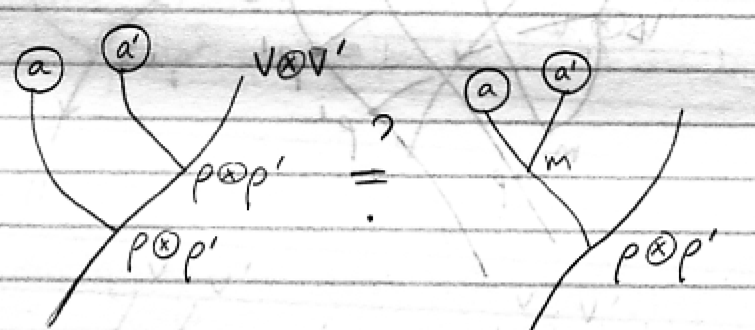
Now — Is $(\rho \otimes \rho')$ a rep.? For this to be true, we need:

① $(\rho \otimes \rho')(aa') = (\rho \otimes \rho')(a)\,(\rho \otimes \rho')(a')$ and

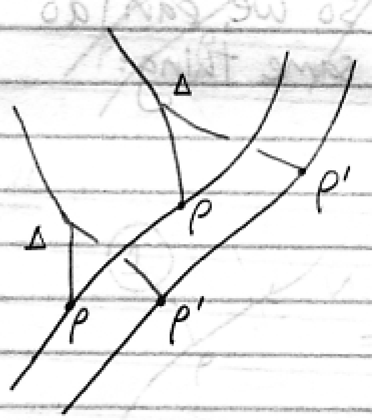
② $(\rho \otimes \rho')(1) = 1_{V \otimes V'}$

① says: (in pictures)

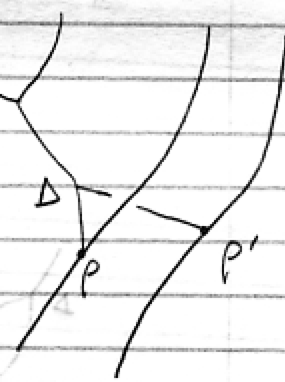
We want this to be equal



LHS:

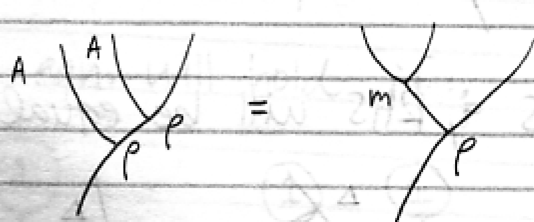


RHS:



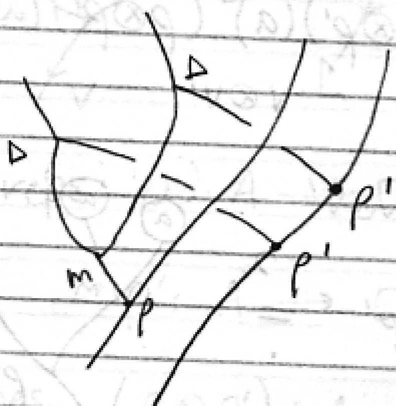
We know ρ is a rep:

ie)

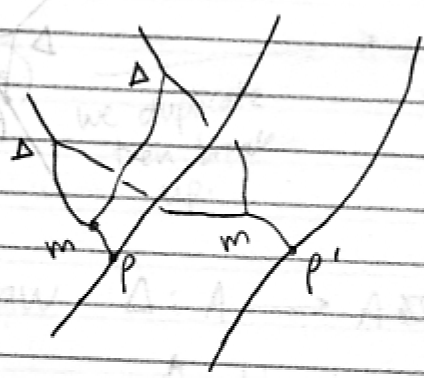


and similarly for ρ' (is a rep).

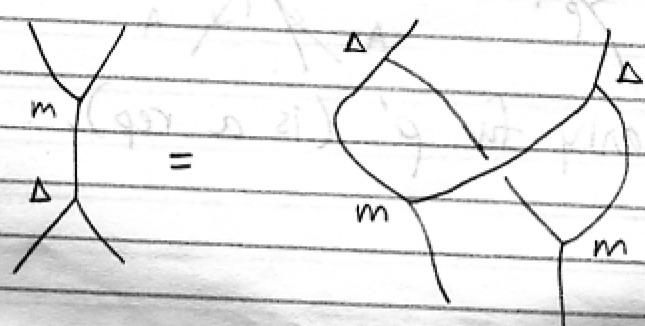
<u>LHS</u>: So we use the fact that ρ is a rep to redraw as:



$\parallel$ since ρ' is also a rep so we can do the same thing:

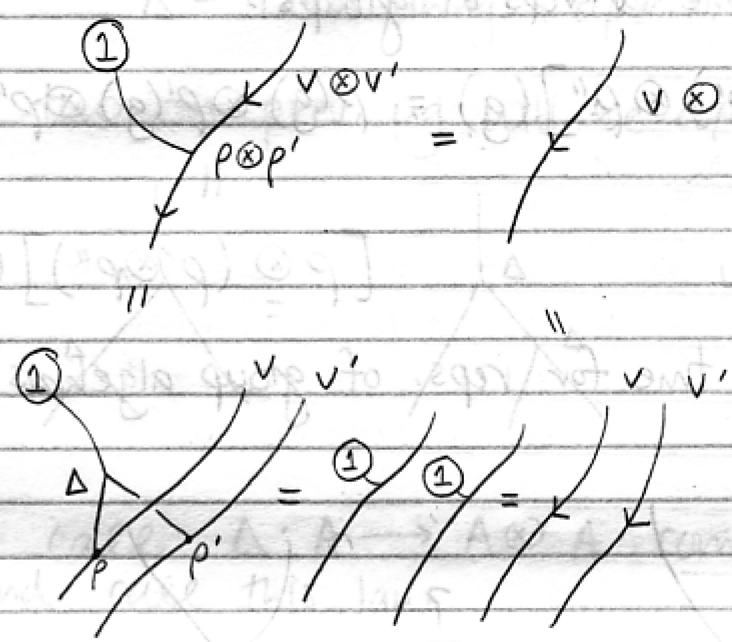


Both the LHS & RHS will be equal if:

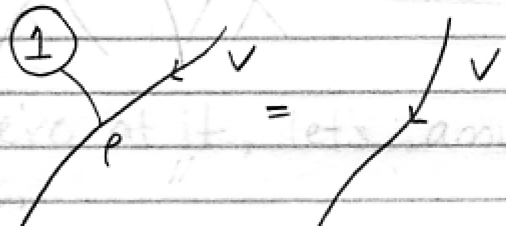


So — let's assume this as one of the axioms for a bialgebra (e.g. quantum group).

we also need: $(\rho \otimes \rho')(1) = 1_{V \otimes V'}$

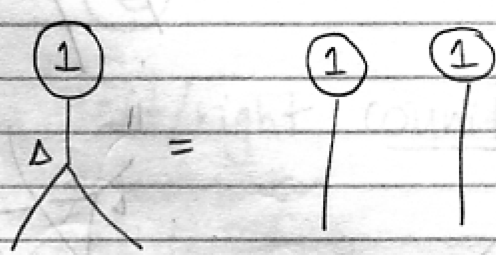


recall:



and similarly for ρ'.

So — the eqn. will hold if:



So we also assume this in defn. of bialgebra.

We'd like to have:

$$(\rho\otimes\rho')\otimes\rho'' = \rho\otimes(\rho'\otimes\rho'')$$

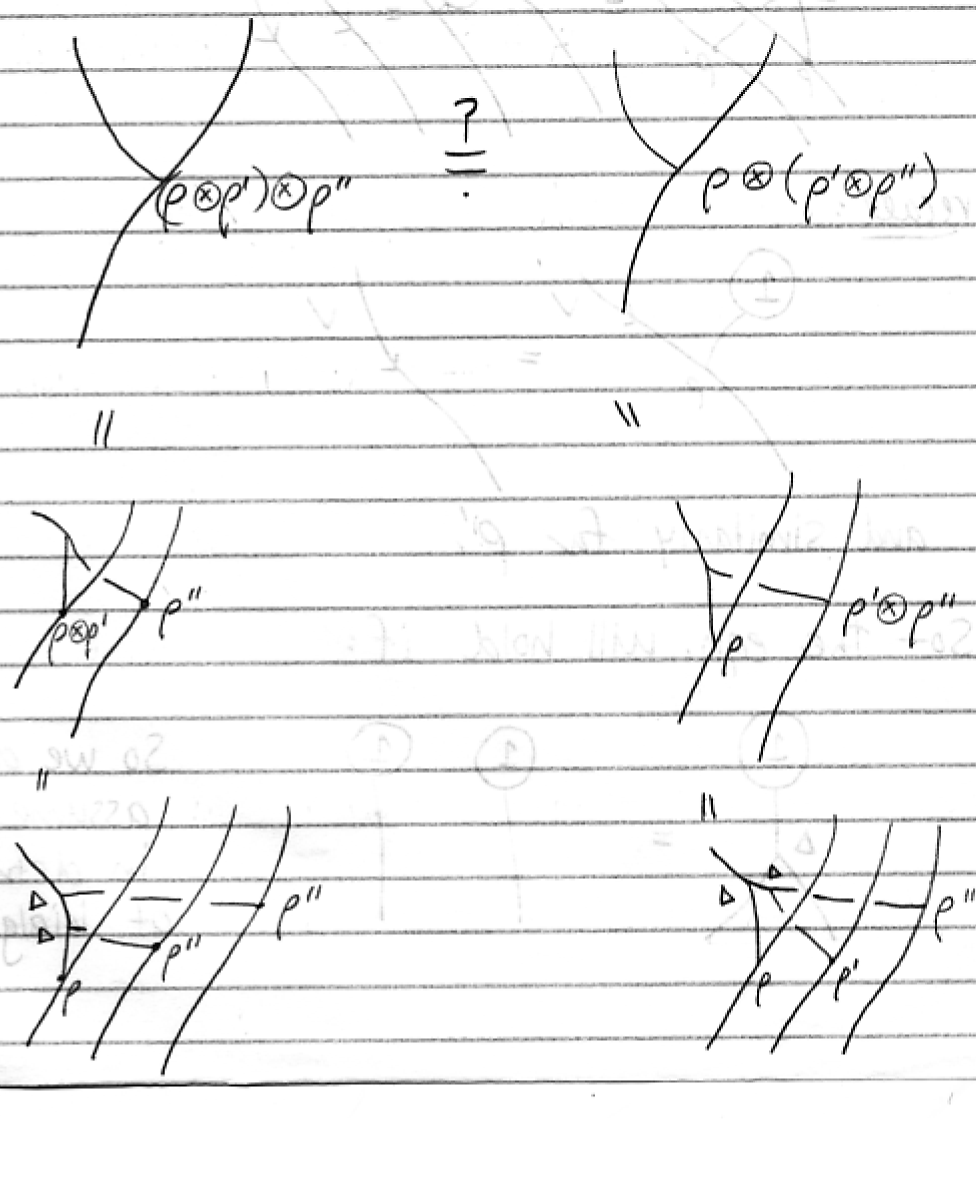
(This is true for reps of groups:

$$[(\rho\otimes\rho')\otimes\rho''](g) = \rho(g)\otimes\rho'(g)\otimes\rho''(g)$$

$$\rho(g)\otimes\rho'(g)\otimes\rho''(g) = [\rho\otimes(\rho'\otimes\rho'')](g)$$

and thus true for reps. of group algebras).

Want:

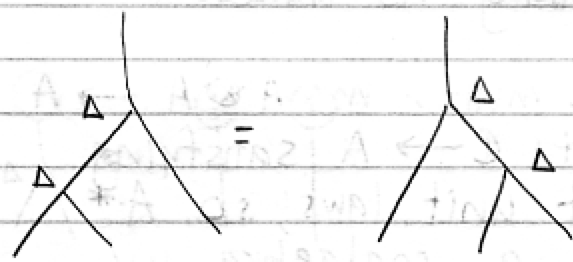


dual of anything is upside-down version

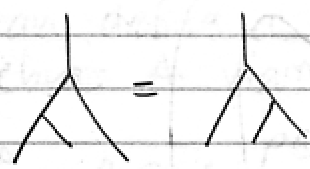
These will be equal if our alg. is assoc. (well- upside-down assoc.)

<u>Note</u>: Δ is "upside-down" mult.
mult — puts things together
Δ - tears things apart.

The two on prev. pg will be equal if:


upside-down associativity.

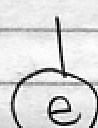
So — call $\Delta : A \longrightarrow A \otimes A$ <u>comultiplication</u> and call this law


<u>coassociativity</u>.

While we're at it, let's assume A has a <u>counit</u>,

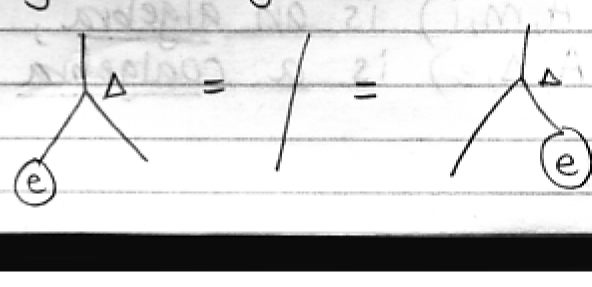
$$e : A \longrightarrow \mathbb{C}$$

we draw as:



we can think of e as an elt of the alg. or a map from $A \rightarrow \mathbb{C}$.

satisfying left/right <u>counit</u> laws:



Defn: A vector space w/ a coassoc., comult. &, counit satisfying the left and right counit laws is called a <u>coalgebra</u>.

exs) of coalgebras:

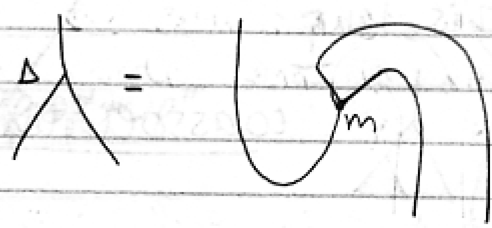
① If A is an <u>algebra</u>, we can form a coalg. by taking its dual.

* turns everything upside down

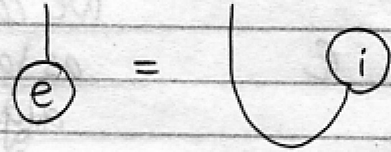
A has assoc. mult. $m: A \otimes A \longrightarrow A$ and unit $i: \mathbb{C} \longrightarrow A$ satisfying left & right unit laws, so A^* will become a coalgebra w/

$\Delta = m^*: A^* \longrightarrow A^* \otimes A^*$

taking duals we can reverse all linear maps!



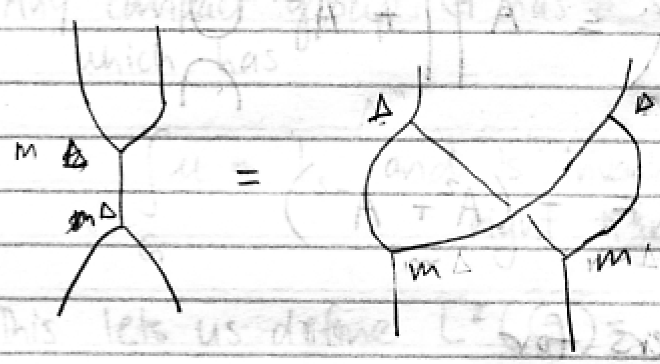
and $e = i^*: A^* \longrightarrow \mathbb{C}$



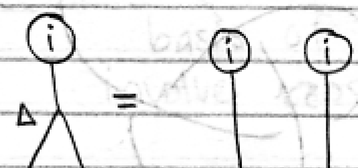
Defn: We define a <u>bialgebra</u> (A, m, i, Δ, e) to be a vector space A st. (A, m, i) is an <u>algebra</u>, (A, Δ, e) is a <u>coalgebra</u>

such that:

comult. is an alg. homo.



and



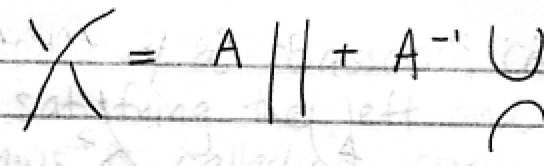
Thm: If A is a bialgebra and ρ, ρ' are reps of A as an algebra, then we can define a rep $\rho \otimes \rho'$ of the algebra A via:

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho \otimes \rho'} \mathrm{End}(V) \otimes \mathrm{End}(V') \cong \mathrm{End}(V \otimes V')$$

and we have $(\rho \otimes \rho') \otimes \rho'' = \rho \otimes (\rho' \otimes \rho'')$

Fact— quantum groups will be bialgebras

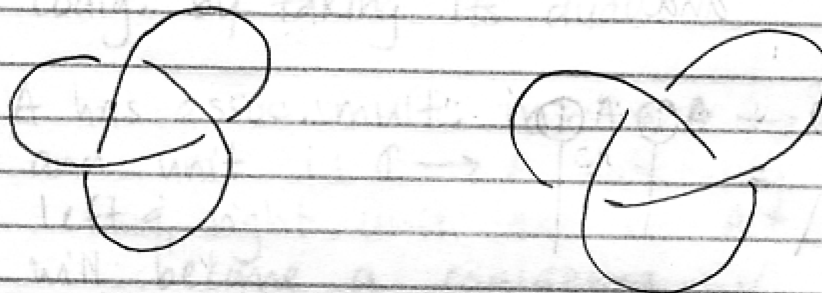
HW#2: Using



$$\bigcirc = -(A^2 + A^{-2})$$

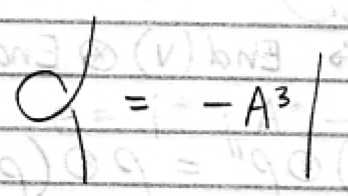
calculate numbers for

Note - fund group can't tell these 2 knots apart.



2 versions of trefoil.

Show that the results are different, and you can't make them the same by multiplying by any power of $-A^3$.



Conclusion: these are different knots.

The number we get by this recipe from any knot (or link) K is called the Kauffman bracket, and denoted: $\langle K \rangle$.

Haar measure

Any compact group G has a unique measure μ which has

$\int_G \mu = 1$ and is invariant under left and right translations and inversion

This lets us define $L^2(G) = L^2(G, \mu)$.

<u>Question</u>: Can we find a nice orthonormal basis of $L^2(G)$? Answer will involve reps. of G.

ex) when G is finite — use δ_{ij} (Kronecker delta).

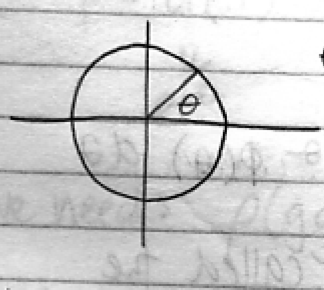
Ex) $G = U(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ unit complex #'s.

is a group under multiplication. (Also called S^1 - the circle.)

Here — we can think of G as $\mathbb{R}/2\pi\mathbb{Z}$

ie) $[0, 2\pi]/\{0 \sim 2\pi\}$ identify 0 & 2π (equivalent)

recall $\mu = dx$ so $d\mu = d\,dx$ doesn't make sense!



$\theta \in [0, 2\pi]/\{0 \sim 2\pi\}$

Here μ has: $\displaystyle\int_{U(1)} f\mu = \int_0^{2\pi} f(\theta)\,\frac{d\theta}{2\pi}$ so $\displaystyle\int_{U(1)} 1\,\mu = 1$

why we divide by 2π

An orthonormal basis of $L^2(U(1))$ is given by:

$$\psi_k(\theta) = e^{ik\theta}$$

Defn. of inner product in L^2 (Hilb. space — so has an inner product) $\langle \psi, \phi \rangle = \int \bar{\psi} \phi$

$$\langle \psi_k, \psi_\ell \rangle = \int_0^{2\pi} \overline{e^{ik\theta}} e^{i\ell\theta} \frac{d\theta}{2\pi}$$

$$= \int_0^{2\pi} e^{i(\ell - k)\theta} \frac{d\theta}{2\pi} = \begin{cases} 1 & \ell = k \\ 0 & \ell \neq k \end{cases}$$

The hard part is showing $\{\psi_k\}$ form a <u>basis</u>.

Showing this uses the Stone-Weierstrass Thm.

So — given any $\phi \in L^2(U(1))$ we get

$$\phi = \sum_{k=-\infty}^{\infty} \hat{\phi}_k \psi_k \quad \text{where}$$

$$\hat{\phi}_k = \langle \psi_k, \phi_k \rangle = \int_0^{2\pi} \bar{\psi}_k(\theta) \phi(\theta) \frac{d\theta}{2\pi}$$

as usual whenever we have an o.n. basis

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \phi(\theta) \, d\theta$$

called the <u>Fourier transform</u> gives $\hat{\phi}_k$ from ϕ.

Fourier transf — write functs as a lin. comb. of $e^{ik\theta}$

The inverse Fourier transform is just:

$$\phi(\theta) = \sum_{k=-\infty}^{\infty} \hat{\phi}_k e^{ik\theta}$$

This allows us to write any funct on the circle to a comb of $e^{ik\theta}$'s which we know how to differentiate, etc.

Question: ① where are these functs. $e^{ik\theta}$ coming from?

② what will we use in their place when we replace $U(1)$ by some other group?

③ How do we do Fourier analysis on a compact group?

Answers: ① They come from the reps (the unitary ones) of $G = U(1)$.

We can get a lot of 1-dim reps of $U(1)$ as follows:

$$\rho : \underset{\substack{\psi \\ e^{i\theta}}}{U(1)} \longrightarrow \mathrm{End}(\mathbb{C}) = \mathbb{C}$$

1x1 Complex matrices (just numbers)

We need: $\rho(gg') = \rho(g)\rho(g')$

$\rho(1) = \underline{1}$

ie) $\rho(e^{i\theta}e^{i\theta'}) = \rho(e^{i(\theta+\theta')}) = \rho(e^{i\theta})\rho(e^{i\theta'})$ and $\rho(1) = 1$.

We can use: raise to a power.

$$\rho(e^{i\theta}) = e^{ik\theta}, \quad k \in \mathbb{Z}$$

Given 2 reps. $\rho : G \to \mathrm{End}(V)$ and
$\rho' : G \to \mathrm{End}(V')$

Then we get: $\rho \oplus \rho' : G \to \mathrm{End}(V \oplus V')$ by

$$\underbrace{(\rho \oplus \rho')(g)}_{\in\, \mathrm{End}(V \oplus V')}(v, v') \overset{\text{Defn}}{=} (\rho(g)v, \rho'(g)v')$$

End$(V \oplus V')$
so need to feed
it an elt of $V \oplus V'$

This is a rep. of G. We say a rep. is <u>irreducible</u> if it's not of the form

$\rho \oplus \rho'$ (unless ρ or ρ' is a rep. on a 0-dim'l vector space)

<u>Thm</u>: If G is a compact group, then every (finite-dim'l) rep. is a direct sum $\rho_1 \oplus \rho_2 \oplus \ldots \oplus \rho_n$ of irreducible reps.

<u>Thm</u>: If G is abelian, all irreducible reps are 1-dim'l.

<u>Note</u>: our group $U(1) = S^1$ is both compact and abelian, so both above results apply.

Thm: Every irreducible rep. of $U(1)$ is (isomorphic to) one of the reps:

$$\rho(e^{i\theta}) = e^{ik\theta} \quad , \quad k \in \mathbb{Z}$$

Note: $e^{ik\theta}$ come from all the irred. reps. of our group $U(1)$.

A generalization of this idea will give us an orthonormal basis of $L^2(G)$ starting from all the irred. reps. of G.
(look at matrix entries – give us the functs).

leading up to Peter-Weyl Thm.