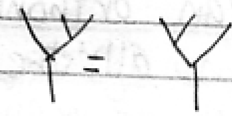
4/17/01

Defn: A <u>bialgebra</u> (V, m, i, Δ, e) is an <u>algebra</u> w/

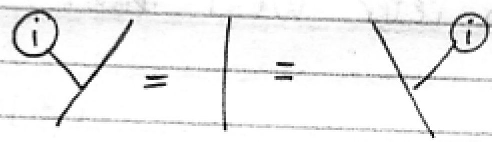
multiplication

and unit

w/ assoc. law

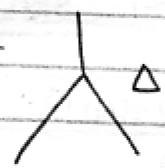


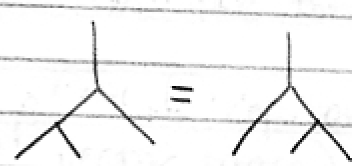
and left & right unit laws



and a <u>coalgebra</u> w/

co-mult



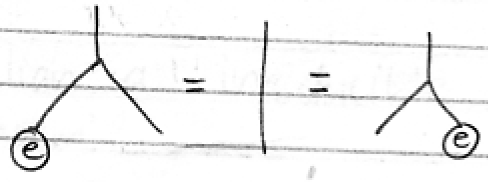


co-assoc.

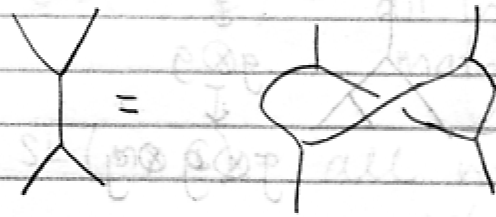
and

counit

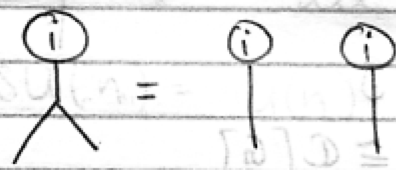
and counit laws:



st these compatibility conditions hold:

① 

and

② 

ex) a group alg is an example of a bialgebra.

<u>Prop</u>: If G is a group, $\mathbb{C}[G]$ is a bialgebra, w/

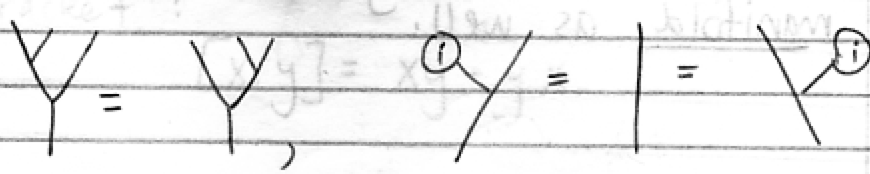
$$m(g \otimes h) = gh \qquad g, h \in G \subseteq \mathbb{C}[G]$$

$$i(1) = 1_G \qquad 1_G \in G.$$

$$\Delta g = g \otimes g$$

$$e(g) = 1$$

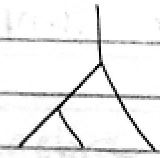
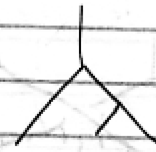
<u>proof</u>: Show assoc, unit laws, coassoc, counit laws.

ie) 

obvious since they hold in the group G.

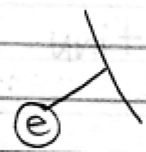
and

$$g \mapsto g \otimes g \mapsto g \otimes g \otimes g$$

 = 

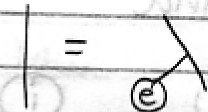
$$g \mapsto g \otimes g \mapsto g \otimes g \otimes g$$

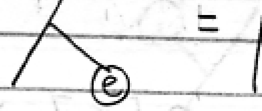
and



$$g \mapsto g \otimes g \mapsto 1 \otimes g \xrightarrow{\sim} g$$

$$1 \otimes g \in \mathbb{C} \otimes \mathbb{C}[G] \cong \mathbb{C}[G]$$

$\Rightarrow$ 

and  similarly.

$*$ <u>HW#3</u>: Check the 2 compatibility conditions.

In addition to getting bialgebras from algebras, we can also get bialgebras from Lie algebras!

<u>Defn</u>: Suppose $G \subseteq M_n(\mathbb{R})$ or $M_n(\mathbb{C})$ or $M_n(\mathbb{H})$ is a group (subset of $n \times n$ real matrices) so closed under matrix mult & inverses. Then G is a <u>Lie group</u> if it is a <u>manifold</u> as well.

ex) $GL(n, \mathbb{C})$ = all invertible complex $n \times n$ matrices.

$SL(n, \mathbb{C})$ = all $n \times n$ complex matrices w/ det = 1.

$U(n, \mathbb{C})$ = all unitary $n \times n$ matrices.
complex conj. transpose = inverse.

$$SU(n) = U(n) \cap SL(n, \mathbb{C})$$

In physics, we often use: $SU(2) \subseteq SL(2, \mathbb{C})$

Given such a (matrix) Lie group, let

$$\mathfrak{g} = \{ X \in M_n \mid e^{tX} \in G \ \forall \ t \in \mathbb{R} \}$$

To exponentiate matrices:

$$e^{tX} = \sum_{n \geq 0}^{\infty} \frac{(tX)^n}{n!} \qquad e^{(s+t)X} = e^{sX} e^{tX}$$

Lie alg - way to study lie groups.
- always a v. space.

Prop: $\mathfrak{g}$ is closed under addition, scalar mult. (so a v. space) and also the "bracket":

$$[X, Y] = XY - YX.$$

This satisfies

① $[\cdot,\cdot]$ is linear in each argument (bilinear)

$$[ax+by, z] = a[x,z] + b[y,z]$$

② $[x,y] = -[y,x]$ anticommutivaty

$$\underset{xy-yx}{[x,y]} = -\underset{(yx-xy)}{[y,x]} \checkmark$$

③ Jacobi identity (due to matrix mult.)

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]].$$

__Defn:__ A __lie algebra__ $\mathfrak{g}$ is a vectorspace w/ $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the above ①–③.

__Digression:__

③ is the product rule in disguise.

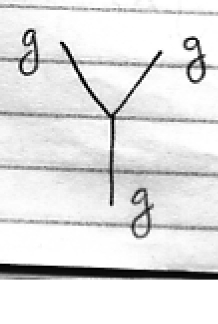
$$d(fg) = f\,d(g) + g\,d(f)$$

where

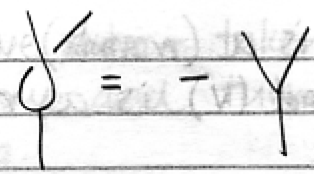
$$d = [x,\cdot]$$

If we draw $[\cdot,\cdot]$ as follows:

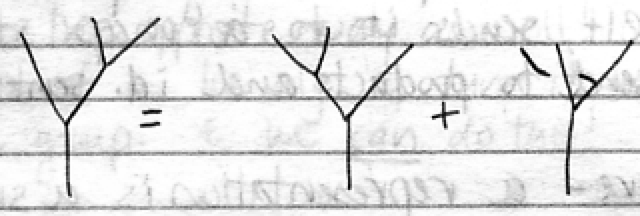
$$[\cdot,\cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \quad \text{as}$$



then:

② says  and

law ③ says



* **HW #3:** Show that the lie algebra of $SL(n,\mathbb{C})$ is $sl(n,\mathbb{C}) = \{ x \in M_n(\mathbb{C}) \mid tr(x) = 0 \}$

ie) Show if $tr(x) = 0$, then e^{tx} has det $= 1$.
& if e^{tx} has det $= 1$, show $tr(x) = 0$.

The Lie algebra of $U(n)$ is $u(n) = \{ x \in M_n(\mathbb{C}) \mid x^* = -x \}$
skew adjoint

Corollary: The Lie alg. of
$SU(n) = SL(n,\mathbb{C}) \cap U(n)$ is
$su(n) = \{ x \in M_n(\mathbb{C}) \mid tr(x) = 0 \ \& \ x^* = -x \}$

Thm: Suppose G is a (matrix) Lie group.
Then if G is connected then any $g \in G$ is a product of elts. e^{tx} where $x \in \mathfrak{g}$ (our lie alg) ($t \in \mathbb{R}$).
If also G is compact, we have $g = e^{tx}$ for some $x \in \mathfrak{g}$, $t \in \mathbb{R}$.

Thm: Suppose G is a (matrix) Lie group and $\rho : G \to \mathrm{End}(V)$ is a representation.

(recall- rep. of a group takes a group elt, sends it to a map st products are send to products and id. sent to identity.

Here- a representation is a smooth funct. st $\rho(gh) = \rho(g)\rho(h)$ and $\rho(1) = 1_V$.
Then we can define
$d\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ by

$$d\rho(x) = \frac{d}{dt} \rho(e^{tx}) \Big|_{t=0} \quad x \in \mathfrak{g}$$

So we can study lie group representations by the lie algebras.

$\mathfrak{g}$, $\mathrm{End}(V)$ are v. spaces

Facts: $d\rho$ is linear, and

$$d\rho([x,y]) = [d\rho(x), d\rho(y)] = d\rho(x)\,d\rho(y) - d\rho(y)\,d\rho(x)$$

$d\rho$ is a representation of a lie algebra.

Defn: A representation of a Lie algebra $\mathfrak{g}$ is a linear map

$$R : \mathfrak{g} \longrightarrow \mathrm{End}(V) \quad \text{st} \quad R([x,y]) = [R(x), R(y)]$$

So – what we've done – taken a representation of a lie group and turned it into a rep. of a lie algebra.

We'd like to go the opposite way as well! ie) take a rep of a lie alg. & turn it into a rep. of a lie group. & we can do this!

Thm: If G is a (matrix) Lie group and G is connected, and $\rho: G \to End(V)$ is a rep. of G, we can recover ρ from $d\rho: g \to End(V)$ via:

recall – G connected means any elt. is a product of e^{tx}.
So, we want to see what ρ does to them.

$$\rho(e^{tx}) = e^{t\, d\rho(x)} \qquad x \in g,\ t \in \mathbb{R}$$

{Thus $\rho(e^{t_1 x_1} \cdots e^{t_n x_n}) = \rho(e^{t x_1}) \cdots \rho(e^{t x_n})$ }.

We want a better result: we want a 1-1 correspondence between ρ's and $d\rho$'s.
To do this, we need:

If G is also simply connected, then given any representation R of g, we get a rep. $\tilde{R}$ of G by:

$$\tilde{R}(e^{tx}) = e^{tR(x)} \qquad t \in \mathbb{R},\ x \in g.$$

<u>Note</u>: $SU(2)$ & $SL(2,\mathbb{C})$ are connected, simply connected.
$SU(2)$ is compact.

what we've done:

1. Given lie group, we can get lie alg.
2. Given rep. of lie group, we can get a rep of a lie alg.
3. If G is "nice" given a rep. of a lie alg, we can get a rep of the lie group.

$$\text{Groups} \longrightarrow \text{Bialgebras}$$
$$G \longmapsto \mathbb{C}[G]$$

$$\text{Lie Groups} \longrightarrow \text{Lie algebras}$$
$$G \longmapsto \mathfrak{g}$$

Now – we'll show how to get:

$$\text{Lie algebra} \longrightarrow \text{Bialgebras}$$
$$\mathfrak{g} \longmapsto U\mathfrak{g}$$

<u>Defn</u>: Given a lie algebra $\mathfrak{g}$ we define the <u>universal</u> <u>enveloping</u> <u>algebra</u> $U\mathfrak{g}$ to be the algebra generated by $\mathfrak{g}$ w/ relations:

$$xy - yx = [x,y] \qquad x, y \in \mathfrak{g}$$

what do we mean by alg. generated by something?

ie) Take all formal linear combs. of formal products of elts. of g. ex) $3x_1x_2x_3 + x_5 + 1$.

and then impose the relations:

① $xy - yx = [x,y]$

② $\underbrace{x+y}_{\text{in } Ug} = \underbrace{x+y}_{\text{addition in the Lie alg, } g}$

③ $\underbrace{ax}_{\text{in } Ug} = \underbrace{ax}_{\text{in } g}$

Note: Ug is a bialgebra.

Thm: Ug is a bialgebra w/ the already given multiplication and unit 1, and comultiplication and counit given by:

Note: $g \subseteq Ug$

$$\Delta : Ug \longrightarrow Ug \otimes Ug$$

$$x \longmapsto x \otimes 1 + 1 \otimes x \qquad x \in g \subseteq Ug$$

and

$$e : Ug \longrightarrow \mathbb{C}$$

$$x \longmapsto 0$$

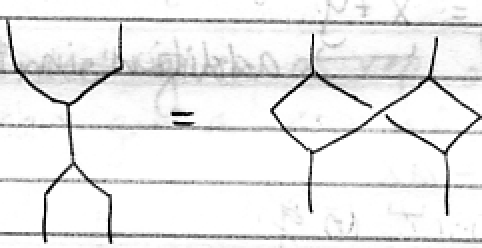
We need to define Δ & e on every elt in Ug, not just elts in g.

Note: The above define Δ & e not just on $g \subseteq Ug$ but on all of Ug, ie. on products $x_1 \cdot \ldots \cdot x_n$ of $x_i \in g$

Eg) $\Delta(xy) = ?$ $\quad x, y \in g$

We use:

$$x \otimes y \mapsto xy \mapsto \Delta(xy)$$



$$x \otimes y \mapsto (x \otimes 1 + 1 \otimes x) \otimes (y \otimes 1 + 1 \otimes y)$$

$$= x \otimes 1 \otimes y \otimes 1 + x \otimes 1 \otimes 1 \otimes y + 1 \otimes x \otimes y \otimes 1 + 1 \otimes x \otimes 1 \otimes y$$

$\mapsto$ now do switching

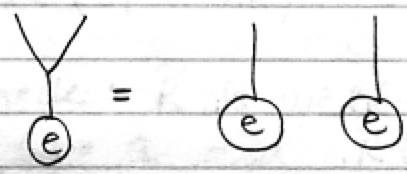
$$x \otimes y \otimes 1 \otimes 1 + x \otimes 1 \otimes 1 \otimes y + 1 \otimes y \otimes x \otimes 1 + 1 \otimes 1 \otimes x \otimes y$$

$$\xmapsto{m \otimes m} xy \otimes 1 + y \otimes x + x \otimes y + 1 \otimes xy$$

So,

$$\Delta(xy) = xy \otimes 1 + y \otimes x + x \otimes y + 1 \otimes xy$$

Similarly we can do $e(x_1 \cdots x_n)$ using:



so we get: $e(1) = 1$ and $e(x_1 \cdots x_n) = 0 \quad n > 0.$

<u>Motivation</u>: for $\mathbb{C}[G]$ we have

$$\Delta g = g \otimes g. \qquad g \in G, \text{ so}$$

If $x \in \mathfrak{g}$ then $e^{tx} \in G$ so that

$$(*) \qquad \Delta e^{tx} = e^{tx} \otimes e^{tx}.$$

We then guess a formula for Δx by differentiating:

$$x = \frac{d}{dt} e^{tx}\Big|_{t=0}$$

so — we differentiate $(*)$ w/r/t t:

$$\frac{d}{dt} \Delta e^{tx}\Big|_{t=0} = \frac{d}{dt} e^{tx} \otimes e^{tx}\Big|_{t=0}$$

Δ is linear, so we get

$$\Delta x = x \otimes 1 + 1 \otimes x$$

$\otimes$ bilinear

Similarly: $e(g) = 1$

$$e(e^{tx}) = 1$$

$$\frac{d}{dt} e(e^{tx}) \Big|_{t=0} = 0$$

$$e(x) = 0.$$

$$\begin{tikzcd} G \arrow[r] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathbb{C}[G] & U\mathfrak{g} \end{tikzcd}$$

G: Lie group; $\mathfrak{g}$: lie alg.

$\mathbb{C}[G]$: group alg. $U\mathfrak{g}$: universal enveloping alg.

Now – want to relate rep. of $\mathfrak{g}$ to rep. of $U\mathfrak{g}$.

We've seen that the reps of G are the same as reps of $\mathbb{C}[G]$.
We've seen that reps. of G are the same as reps of $\mathfrak{g}$, if G is connected and simply connected.

Now: reps of $\mathfrak{g}$ are the same as reps of $U\mathfrak{g}$.

Given a rep R of $\mathfrak{g}$:

$R: \mathfrak{g} \longrightarrow \mathrm{End}(V)$ then we define a rep:

$\tilde{R}: U\mathfrak{g} \longrightarrow \mathrm{End}(V)$ by:

$$\tilde{R}(x_1 \cdots x_n) = R(x_1) \cdots R(x_n) \quad x_i \in g$$

Can check: $\tilde{R}$ is well-defined and a rep.

$\tilde{R}(1) = 1$ and $\tilde{R}(ab) = \tilde{R}(a)\tilde{R}(b) \quad a, b \in Ug$

Conversely – given a rep $\tilde{R}: Ug \longrightarrow End(V)$ just let

$R: g \longrightarrow End(V)$ be $R(x) = \tilde{R}(x)$
$\forall x \in g \subseteq Ug$

Check it's a representation.

$\hbar$: Planck's constant

when $\hbar = 0$, we have classical mechanics.
But as we turn on $\hbar$ (make it nonzero)
Ug will turn into the "quantum group" $U_\hbar g$
which is still a bialgebra.

We'll do it for $g = su(2)$ or $sl(2, \mathbb{C})$

Track 2: <u>Goal</u>: If G is a compact group, we want a description of $L^2(G)$ — an orthonormal basis of it.

If $G = U(1)$ this basis is just

$$\psi_k(x) = e^{ikx}.$$

These come from irreducible reps of $U(1)$:

$$\rho_k : U(1) \longrightarrow \mathrm{End}(\mathbb{C})$$

$$e^{ix} \longmapsto e^{ikx}$$

$$\rho_k(e^{ix}e^{iy}) = \rho_k(e^{ix})\,\rho_k(e^{iy}) \checkmark$$

$$\rho_k(1) = 1.$$

Given a rep of G, we get functs on G:

In general — Given a rep. $\rho : G \longrightarrow \mathrm{End}(V)$ how can we get functions on G?

If $V = \mathbb{C}^n$, then $\mathrm{End}(V) = M_n(\mathbb{C})$, $n \times n$ complex matrices.

Thus $\rho(g)$ is an $n\times n$ matrix and each entry $\rho_{ij}(g)$ is a complex number.

$$\rho_{ij} : G \longrightarrow \mathbb{C} \qquad 1 \le i,j \le n.$$

(the entries of our matrix give us functs!)

Thm: If we let ρ range over all different irreducible reps. of G, the functions $\rho_{ij}: G \to \mathbb{C}$ (that we get from G) we make them orthonormal (Gram-Schmidt) and they form an o.n. basis of $L^2(G)$.

We'll prove this slowly.

ie G is a compact top. space

Given a compact group G, a representation $\rho: G \to \mathrm{End}(V)$ is a continuous funct. st

$$\rho(gh) = \rho(g)\rho(h) \quad \text{and} \quad \rho(1) = 1.$$

If ρ is cont $\Rightarrow \rho_{ij}: G \to \mathbb{C}$ are continuous. In fact if G is a lie group, ρ_{ij} are smooth — that is, infinitely differentiable.

Given reps $\rho: G \to \mathrm{End}(V)$, $\rho': G \to \mathrm{End}(V')$ we get reps

$$\rho \oplus \rho' : G \longrightarrow \mathrm{End}(V \oplus V')$$

$$\rho \otimes \rho' : G \longrightarrow \mathrm{End}(V \otimes V')$$

st

$$(\rho \oplus \rho')(g)\,(v, v') = (\rho(g)v, \rho'(g)v')$$

and

$$(\rho \otimes \rho')(g)\; v \otimes v' = \rho(g)v \otimes \rho'(g)v'$$

We now want to talk about the morphisms bet. representations.

Defn: An <u>intertwining operator</u> (or <u>intertwiner</u>) from ρ to ρ' is a linear operator $T: V \to V'$ st

$$T\rho(g)v = \rho'(g)Tv \quad \text{"intertwining condition"}$$

equivalently:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\rho(g)"'] \arrow[dr, phantom, "\circlearrowright"] & V' \arrow[d, "\rho'(g)"] \\ V \arrow[r, "T"'] & V' \end{tikzcd}$$

this diagram commutes.

Given intertwiners $T: V \to V'$ and $T': V' \to V''$ we can compose them: $T'T: V \to V''$.

<u>pf</u>: Show $T'T$ is an intertwiner:

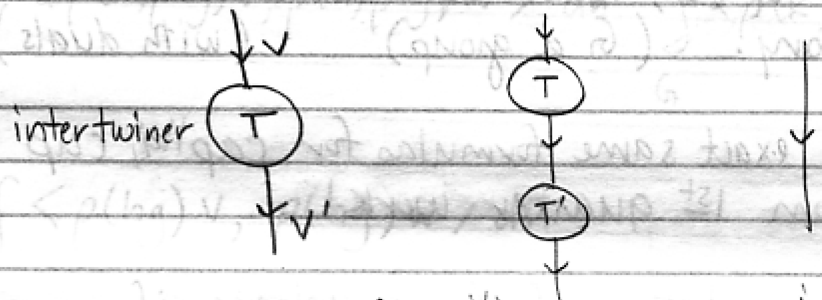
$$T'T\rho(g)v = T'\rho'(g)Tv = \rho''(g)T'Tv \quad \checkmark$$

Any <u>rep</u> $\rho: G \to \mathrm{End}(V)$ has an <u>identity</u> intertwiner:

$$1_V : V \to V$$

$$\rho(g)1_V v = 1_V \rho(g) v. \quad \checkmark$$

So — we get a <u>category</u> Rep(G), whose objects are the reps of G and morphisms are intertwiners.



composition of intertwiners

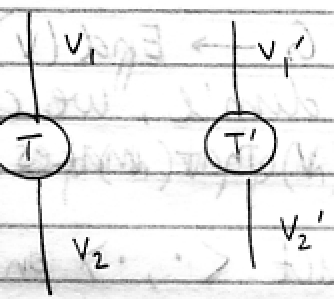
identity intertwiner

* Our picture tricks work in Rep(G).

Given reps $\rho: G \longrightarrow \mathrm{End}(V)$ and $\rho': G \longrightarrow \mathrm{End}(V')$ we draw

$$\rho \otimes \rho' : G \longrightarrow \mathrm{End}(V \otimes V')$$

by horizontal setting side-by-side

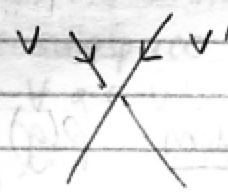


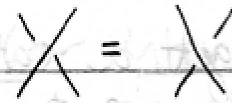
denotes the intertwiner

$$T \otimes T' : V_1 \otimes V_1' \longrightarrow V_2 \otimes V_2'$$

There are <u>braiding</u> intertwiners:

$$B_{V,V'} : V \otimes V' \longrightarrow V' \otimes V$$
$$V \otimes W \longmapsto W \otimes V$$



Also — $B_{V,V'} = B_{V',V}^{-1}$: 

So — Rep(G) is a symmetric monoidal category. (G a group) (with duals)

Note — exact same formulas for cap & cup from 1st quarter work!

Defn: A representation of G is unitary if

① V is a Hilbert space

② $\rho : G \longrightarrow \operatorname{End} V$ is st $\rho(g) : V \longrightarrow V$ is unitary $\forall g \in G$.

If G is compact, we can turn any representation into a unitary one:

Given any rep $\rho : G \longrightarrow \operatorname{End}(V)$ where V is finite-dim'l, we can find an inner product on V that makes ρ unitary.

Recall — compact group has Haar measure on it.

Take any inner product $\langle \cdot , \cdot \rangle$ on V and let

$$\langle\langle x, y \rangle\rangle = \int_G \langle \rho(g) v, \rho(g) w \rangle \, dg$$

Claim: $\rho(g)$ is unitary w/r/t this new inner product $\langle\langle \cdot , \cdot \rangle\rangle$

Check: $\langle\langle \rho(g) v, \rho(g) w \rangle\rangle = \langle\langle v, w \rangle\rangle$

Check: $\langle\langle \rho(g)v, \rho(g)w \rangle\rangle \stackrel{?}{=} \langle\langle v, w \rangle\rangle$

$$\int_G \langle \rho(h)\rho(g)v, \rho(h)\rho(g)w \rangle \, dh \qquad \int_G \langle \rho(k)v, \rho(k)w \rangle \, dk$$

ρ is a rep. so we can do this!

$$= \int_G \langle \rho(hg)v, \rho(hg)w \rangle \, dh$$

Now want to use property of Haar measure:

$$\int f(hg)\, dh = \int f(h)\, dh$$

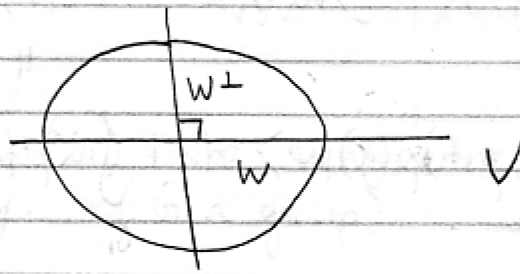
So above $\int_G \langle \rho(hg)v, \rho(hg)w \rangle \, dh$

$$= \int_G \langle \rho(h)v, \rho(h)w \rangle \, dh \quad \checkmark \text{ equal to RHS.}$$

<u>Thm</u>: If $\rho: G \longrightarrow \mathrm{End}(V)$ and it's unitary, and $W \subseteq V$ is a vector subspace and W has $\rho(g): W \longrightarrow W \ \forall g \in W$. (we call W a <u>subrepresentation</u>). Then,

$W^\perp$ (the orthogonal complement — set of all vectors $\perp$ to W)

is <u>also</u> a subrepresentation.



Then we say $V = W \otimes W^\perp$.

<u>proof</u>: Need to show $\rho(g) : W^\perp \longrightarrow W^\perp$,
ie)

$$\langle x, w \rangle = 0 \;\; \forall\, w \in W, \text{ then}$$

$$\Rightarrow \langle \rho(g)x, w \rangle = 0 \;\; \forall\, w \in W.$$

$$\langle \rho(g)x, w \rangle = \langle \rho(g^{-1})\rho(g)x, \rho(g^{-1})w \rangle$$
$$= \langle x, \underbrace{\rho(g^{-1})w}_{\in W} \rangle$$

But x is $\perp$ to everything in W

$$\Rightarrow \quad \langle x, \rho(g^{-1})w \rangle = 0.$$

$\boxtimes$

So – <u>Corollary</u>: If G is compact, we can break up any rep into a direct sum of irreducible reps.