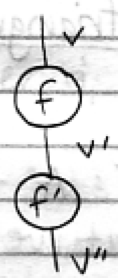
4/24/01

If A is an <u>algebra</u> there is a category whose objects are representations of A and whose morphisms are intertwiners:

if $\rho : A \longrightarrow \mathrm{End}(V)$ are reps, an
$\rho' : A \longrightarrow \mathrm{End}(V')$

<u>intertwiner</u> is a linear map $f : V \longrightarrow V'$ st.
$$f\rho(a)v = \rho'(a)fv$$

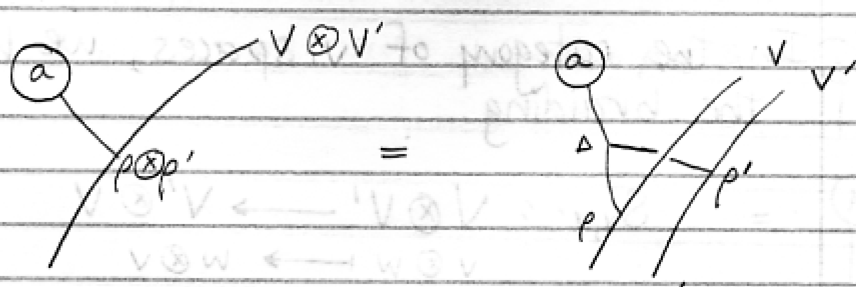
<u>Note</u>: composition of intertwiners is an intertwiner



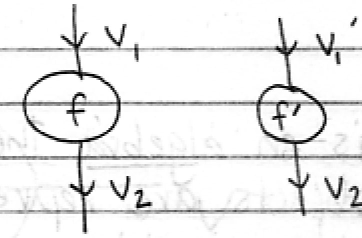
If A is a <u>bialgebra</u> this (category we mentioned above) is a <u>monoidal category</u>:

we can tensor reps $\rho : A \longrightarrow \mathrm{End}(V)$
$\rho' : A \longrightarrow \mathrm{End}(V')$
to get
$$\rho \otimes \rho' : A \longrightarrow \mathrm{End}(V \otimes V').$$

via:



f, f' intertwiners:



If $f: V_1 \longrightarrow V_2$, $f': V_1' \longrightarrow V_2'$ are intertwiners so is $f \otimes f': V_1 \otimes V_2 \longrightarrow V_1' \otimes V_2'$, which we draw as above.

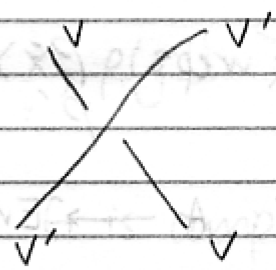
What extra structure must a bialgebra have for its category of reps to be a <u>braided</u> monoidal category.

We'll call such a bialgebra <u>quasitriangular</u> (or <u>braided</u>).

Given 2 reps $\rho: A \longrightarrow \mathrm{End}(V)$, $\rho': A \longrightarrow \mathrm{End}(V')$ of a bialgebra A. Let's see what we need to define the braiding:

$$B_{V,V'}: V \otimes V' \longrightarrow V' \otimes V$$

$B_{V,V'}$ will be drawn as:



In the category of v. spaces, we had the braiding

$$S_{V,V'}: V \otimes V' \longrightarrow V' \otimes V$$
$$v \otimes w \longmapsto w \otimes v$$

Note – This is what we use for reps of a group.

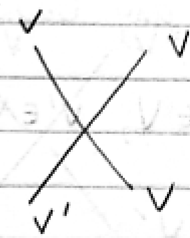
Sometimes we can just use this for our $B_{v,v'}$

e.g.)

$A = \mathbb{C}[G] \quad U_g$

We secretly were doing this in the 1^{st} quarter w/ $G = SU(2)$ or $SL(2, \mathbb{C})$.

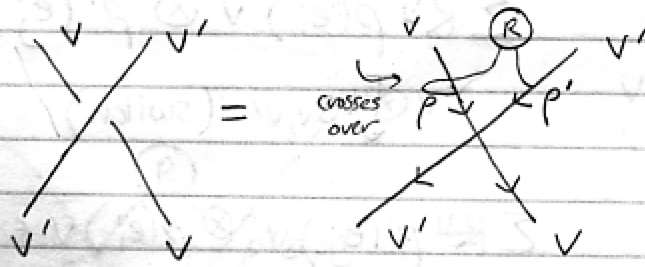
We want to redraw our original braiding, $S_{v,v'}$ as:



This agrees w/ the fact that  when we use $S_{v,v'}$. ie) it doesn't matter which one is over & which one is under crossing.

Now, let's cook up an interesting $B_{v,v'}$ as follows: (we want to define the new braiding in terms of the old one).

new braiding:

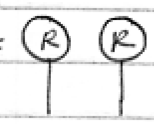


$R \in A \otimes A$

"an R-matrix"

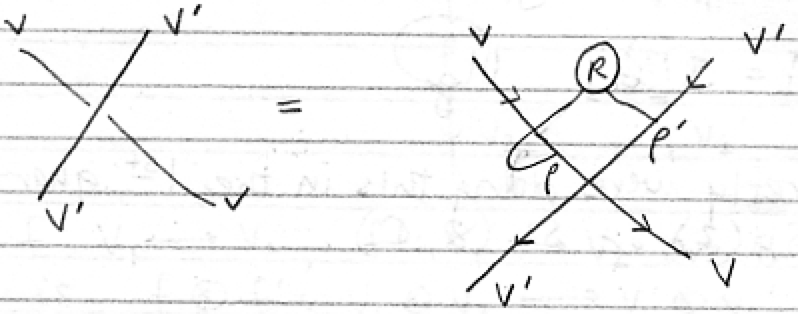
Recall —

$R \in A =$ 

$R \in A \otimes A =$ 

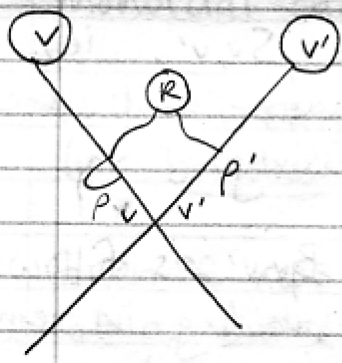
$=$ 

Let's write our pictures in eqns:



$R = \sum R^{ij} e_i \otimes e_j$ $\quad$ e_i any basis for A

$v \in V, \; v' \in V'$

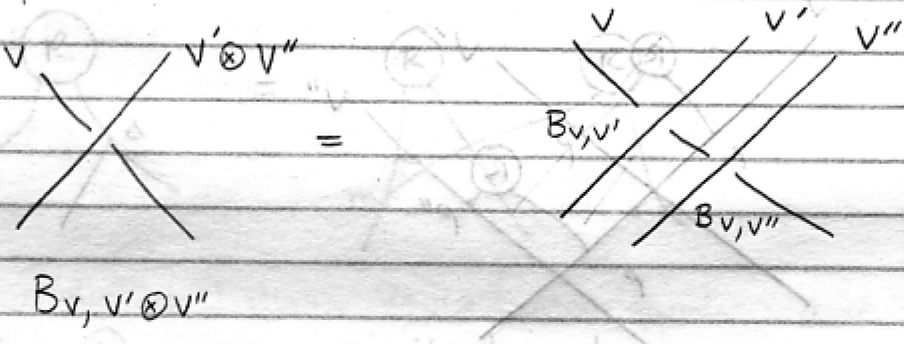


$$
\begin{aligned}
& v \otimes v' \\
& \quad \downarrow \\
& v \otimes R \otimes v' \\
& \quad \parallel \\
& \sum R^{ij}\, v \otimes e_i \otimes e_j \otimes v' \\
& \quad \downarrow \\
& \sum R^{ij} \rho(e_i) v \otimes \rho'(e_j) v' \\
& \quad \downarrow S_{V,V'} \text{ (switch)} \\
& \sum R^{ij} \rho'(e_j) v' \otimes \rho(e_i) v \in V' \otimes V
\end{aligned}
$$

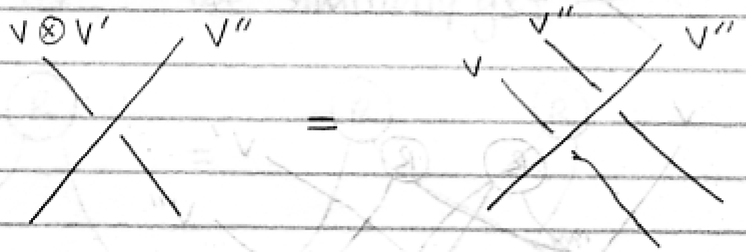
Recall – the braiding last quarter had to satisfy certain properties, and so our braiding here will have to as well.

(from last term)

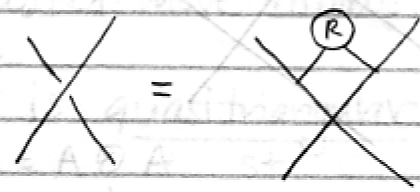
The defn. of a braided monoidal category requires:

① 

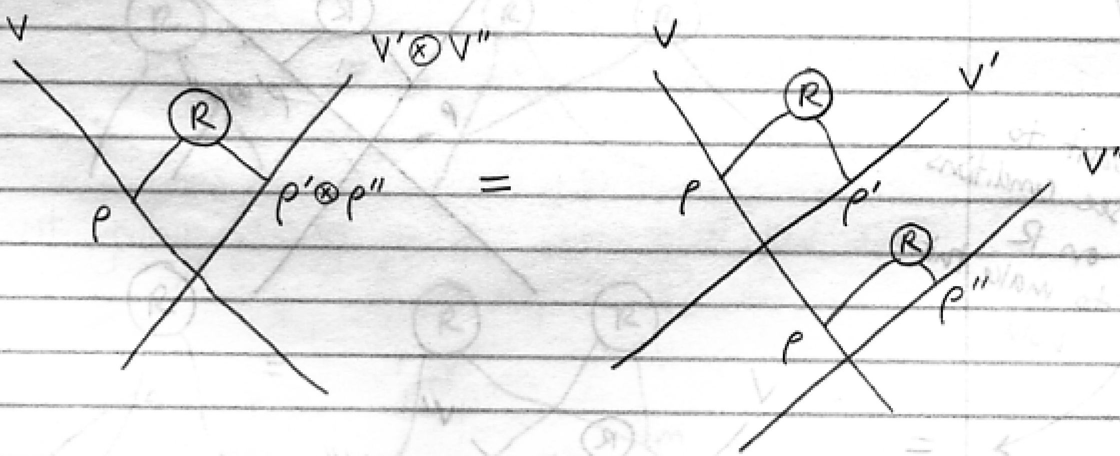
And

② 

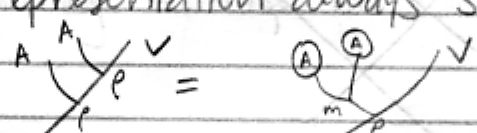
These are going to give us conditions R must satisfy.

Imposing ① on 

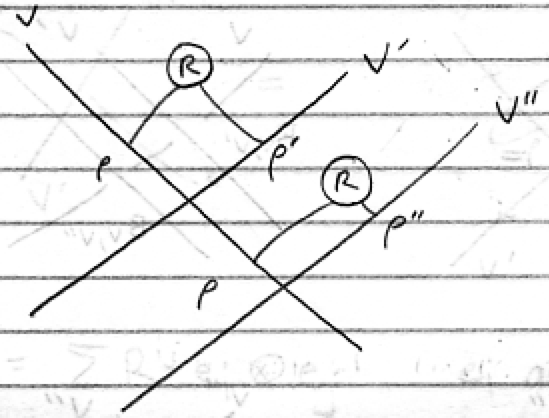
we get



Recall, a representation always satisfies



So, RHS becomes



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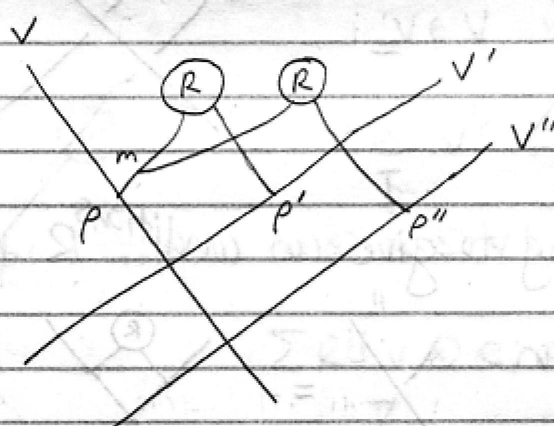
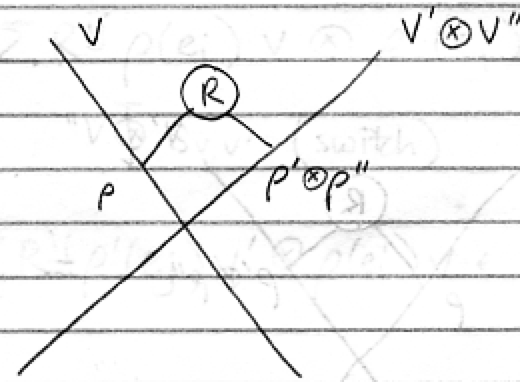
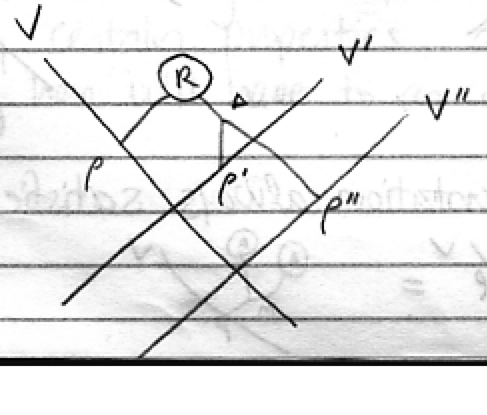


and LHS

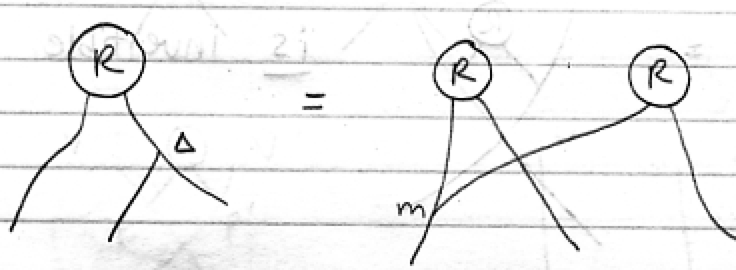


want to see conditions on R to make this hold.

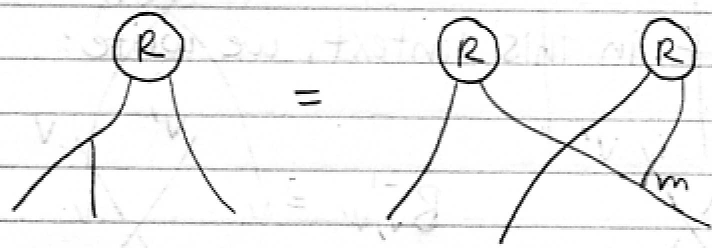
=



So — this will hold if



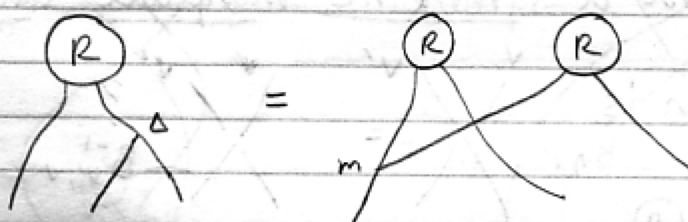
Imposing ② — we similarly get —



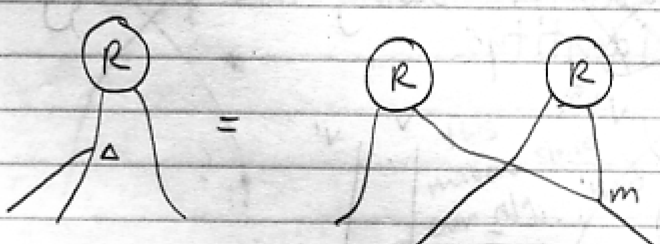
Note — 3rd Reidemeister move follows from ① & ②.

So — a bialgebra A is <u>quasitriangular</u> if it is equipped w/ $R \in A \otimes A$ st

①



②

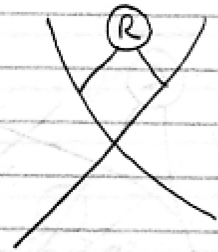


Also —

we want braiding to have an inverse.

And — in fact

$B_{v,v'}$ =

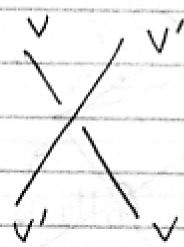


is invertible.

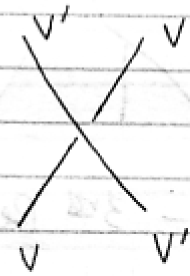
(And perhaps more stuff needed to get a braided monoidal category.)

Henceforth — in this context, we write:

$B_{v,v'}$ =

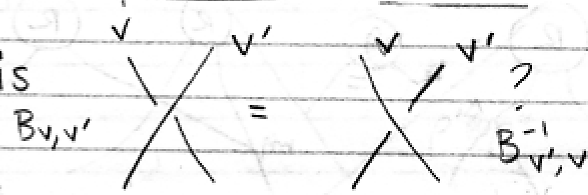


$B^{-1}_{v,v'}$ =



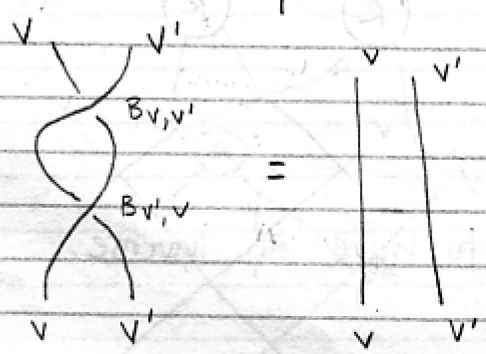
When is the category of representations of a quasitriangular bialgebra a symmetric monoidal category.

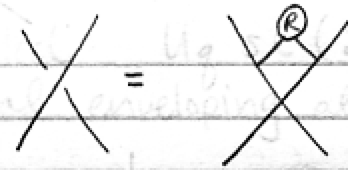
ie) when is

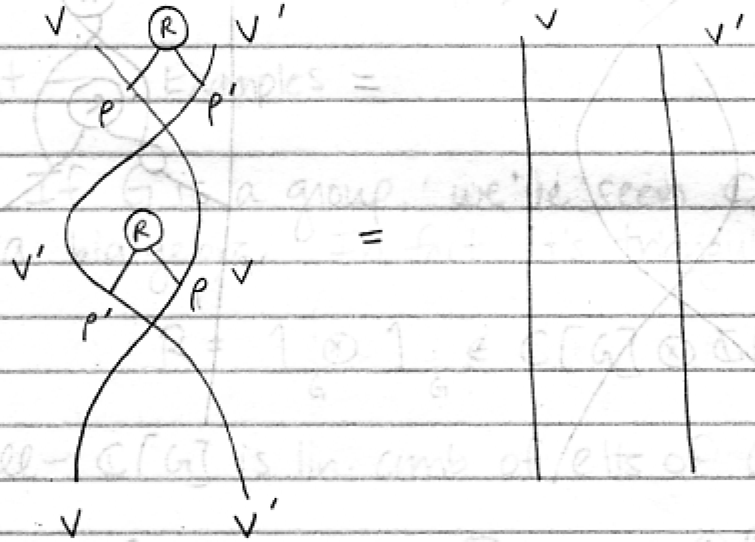


This will hold iff doing the braiding twice is the identity

ie)



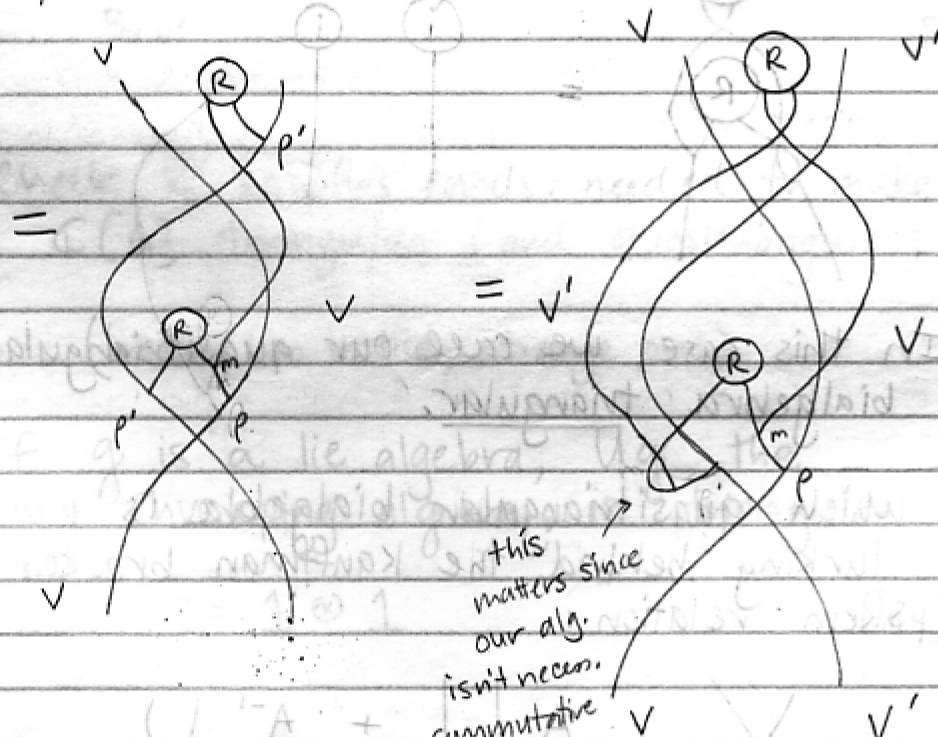
Using  this says



Do same thing to LHS as before — use fact that ρ, ρ' are reps.

Using fact ρ is a rep:

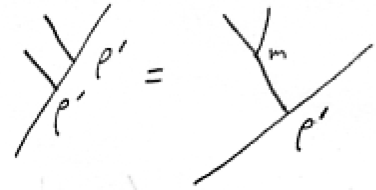
S = usual braiding, usual way of switching 2 vectors



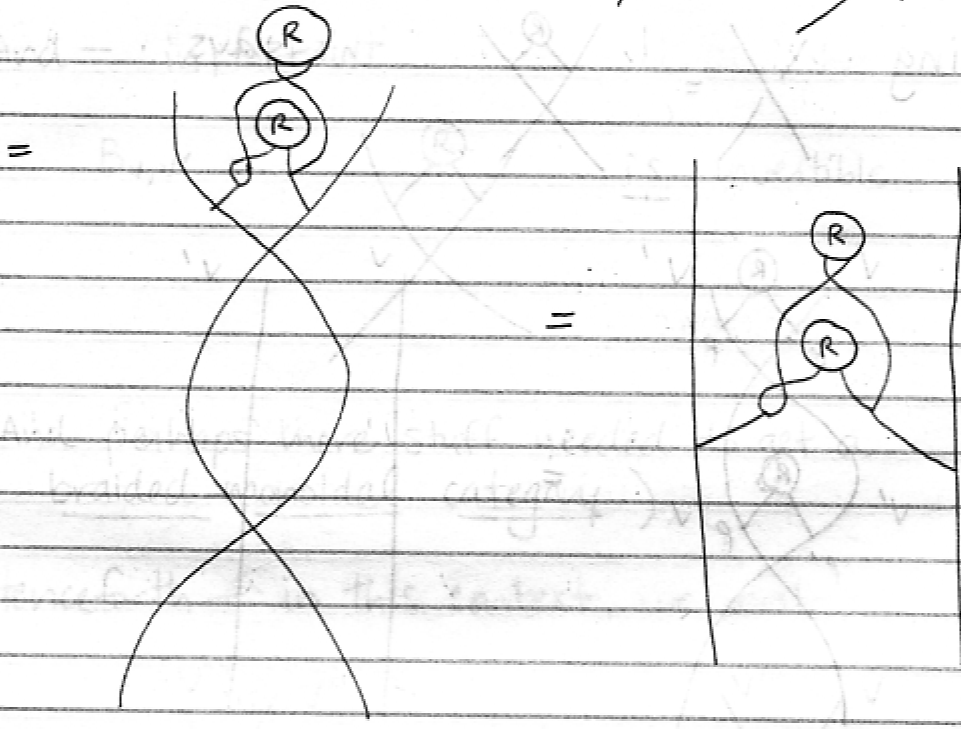
since ρ' is a rep.

this matters since our alg. isn't necess. commutative

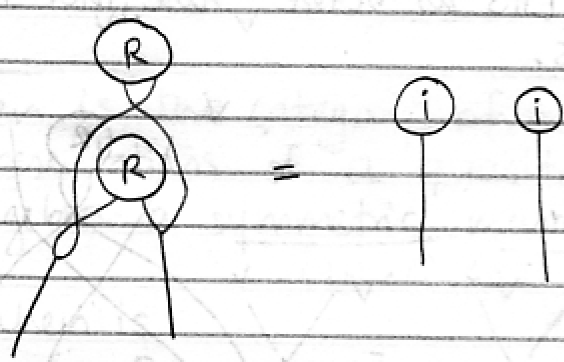
Fact about reps that we're using:



from prev pg.



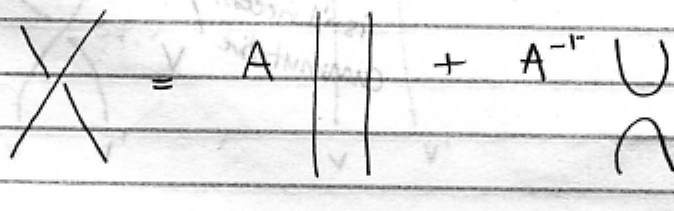
So – the braiding is a symmetry if



In this case, we call our quasitriangular bialgebra <u>triangular</u>.

---

which quasitriangular bialgebra is lurking behind the Kauffman bracket skein relation:



or "q-deformed"

It's called: $U_q\, sl(2,\mathbb{C})$ – the "quantized" universal enveloping algebra of $sl(2,\mathbb{C})$.

(A is some function of q.)

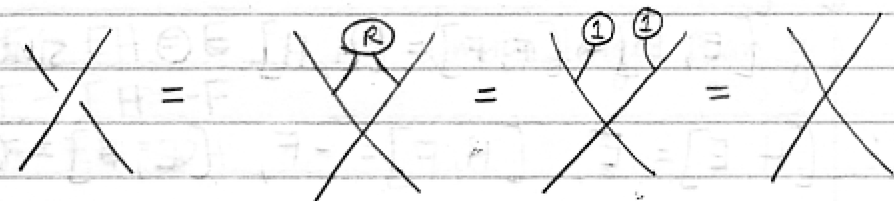
First — Examples

① Ex) If G is a group, we've seen $\mathbb{C}[G]$ is a bialgebra. In fact, it's <u>triangular</u> w/

$$R = 1_G \otimes 1_G \in \mathbb{C}[G] \otimes \mathbb{C}[G].$$

Recall – $\mathbb{C}[G]$ is lin. comb of elts of G.

so —



$B_{v,v'}$ $\qquad\qquad\qquad\qquad$ $S_{v,v'}$

<u>HW #4</u>: Check R satisfies conds. needed to make $\mathbb{C}[G]$ triangular (and quasitriangular)

(braiding has inverse, strange one)

② If g is a lie algebra, Ug, the univ. enveloping algebra, is triangular w/

$$R = 1 \otimes 1$$

$[x,y] = xy - yx$

If $\mathfrak{g} = sl(2, \mathbb{C})$ what's $U_q\mathfrak{g}$ like?

$sl(2,\mathbb{C}) = \{2 \times 2$ traceless $(tr(x)=0)$ complex matrices$\}$

(4-dim'l since 4 entries, but $tr=0$ condition knocks us down to 3-dim'l.

Our basis for $sl(2, \mathbb{C})$ is:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

These satisfy

$$[E,E] = [F,F] = [H,H] = 0 \quad \text{since } [x,y] = xy - yx$$

HW#4: $[H,E] = E, \quad [H,F] = -F, \quad [E,F] = 2H$

physics: H: "angular momentum along z-axis" (observable)

E: "raising operator"
F: "lower operator"
} have the effect of raising or lowering ang. momentum.

If $\psi_\uparrow = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2$ then $H\psi_\uparrow = \frac{1}{2}\psi_\uparrow$

"spin-up" state

So $\psi_\uparrow$ is a state where angular momentum along the z-axis is $+\frac{1}{2}$.

electron – spin $\frac{1}{2}$ particle

If $\psi_\downarrow = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$ then $H\psi_\downarrow = \frac{-1}{2}\psi_\downarrow$

$\psi_\downarrow$ is "spin down".

<u>Note</u>: $E\psi_\downarrow = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \psi_\uparrow$

so that E is a <u>raising</u> operator. Note: $E\psi_\uparrow = 0$.

<u>Similarly</u>, F deserves the name <u>lowering</u> operator.

So — if $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, $U\mathfrak{g}$ is the <u>associative algebra</u> generated by E, F & H modulo the relations

univ. enveloping alg. (much larger than $\mathfrak{g}$).

$$HE - EH = E$$
$$HF - FH = -F$$
$$EF - FE = 2H$$

and it's a <u>bialgebra</u>, (recall — we can make any univ. enveloping alg into a bialgebra by defining Δ).

with
$$\Delta E = E \otimes 1 + 1 \otimes E$$
$$\Delta F = F \otimes 1 + 1 \otimes F$$
$$\Delta H = H \otimes 1 + 1 \otimes H$$

and it's <u>triangular</u> w/ $R = 1 \otimes 1$.

$U_q\, sl(2,\mathbb{C})$ is similar but it's an algebra generated by $E, F, K =$ "q^H", K^{-1}

$$= \text{"}e^{\hbar H}\text{"}$$

(where $q = e^{\hbar}$ is a number)

We'll get back $U sl(2,\mathbb{C})$ in some sense as $\hbar \to 0$ or $q \to 1$. univ. envelop. alg.

$*$ $KE = qEK$

$KF = q^{-1}FK$

$*$ $EF - FE = \dfrac{K^2 - K^{-2}}{q - q^{-1}}$

$U_q\, sl(2,\mathbb{C})$ becomes a bialgebra w/

$\Delta E = E \otimes K + K^{-1} \otimes E$

$*$ $\Delta F = F \otimes K^{-1} + K \otimes F$

$\Delta K = K \otimes K$

We'll check $*$

① The $\hbar \to 0$ $(q \to 1)$ limit:

$$K = e^{\hbar H} = \underbrace{1 + \hbar H}_{\text{first-order terms}} + \frac{(\hbar H)^2}{2!} + \dots$$

So, $KE = qEK$ says

$$e^{\hbar H} E = e^{\hbar} E e^{\hbar H}$$ expand out exponential

$$(1 + \hbar H + \dots) E = (1 + \hbar + \dots) E (1 + \hbar H + \dots)$$

$$E + \hbar H E + \dots = E + \hbar E + \hbar E H + \dots$$

$\Rightarrow$ $E = E$ ✓ and

$HE - EH = E$ ✓ (bring $\hbar EH$ to LHS)
(which we know is true!)

② $$EF - FE = \frac{e^{2\hbar H} - e^{-2\hbar H}}{e^{\hbar} - e^{-\hbar}}$$

only 1st order terms $$= \frac{4\hbar H}{\hbar - (-\hbar) = 2\hbar} = 2H$$ ✓ (what we have in our relations)

③ $$\Delta F = F \otimes K^{-1} + K \otimes F$$

$$= F \otimes e^{-\hbar H} + e^{\hbar H} \otimes F$$

lowest (zeroth) order terms $= F \otimes 1 + 1 \otimes F$ (which is what we have)

Track 2: G - compact group

Some facts:

① If $\rho: G \longrightarrow End(V)$ is a rep. then we can find an inner product on V st ρ is unitary ie)

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle \quad \forall \; v, w \in V, \; g \in G$$

② If $\rho: G \longrightarrow End(V)$ is unitary & $W \subseteq V$ is a subrep - ie $\rho(g): W \longrightarrow W$ $\forall g \in G$.

Then $W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \; \forall w \in W\}$
(all vectors $\perp$ to W)
is also a subrep.
Then V is a direct sum of the reps $W, W^\perp$: $V = W \oplus W^\perp$ as reps.

Note — we don't think of $\{0\}$ as irreducible just as we don't think of 1 as being prime.

ex) of subreps: $0 \subseteq V$ & $V \subseteq V$.

③ We say V is irreducible if $\{0\}$ and V are the only subreps of V.
(Note - it's evil to say $\{0\}$ is irreducible)

Any rep V is a finite direct sum of irreducible reps
$$V = \bigoplus_{i=1}^{n} W_i$$

(In fact, there is essentially a unique way to do this.)

Schur's lemma (Part I): Suppose

$\rho: G \longrightarrow End(V)$ and $\rho': G \longrightarrow End(V')$ are two irreducible reps of G.

Suppose $f: V \longrightarrow V'$ is an intertwiner:

ie) $f\rho(g)v = \rho'(g)fv \quad \forall g \in G, \forall v \in V.$

dicotomy: Then either:

(1) f is 1-1 & onto (so it's invertible, so our reps are isomorphic)

or

(2) $f = 0$.

proof: Look at $\ker f = \{v \in V \mid f(v) = 0\} \subseteq V$

$\text{range} f = \{v' \in V' \mid v' = f(v) \text{ for some } v \in V\} \subseteq V'$

Claim: These are subreps of V, V' respectively.

But V, V' are irreducible, meaning the only subreps of them are $\{0\}$ or V (or V').

$\Rightarrow$ $\ker f$ is either $\{0\}$ or V.

range f is either $\{0\}$ or V'.

If $\ker f = V$, everything is sent to zero, so $f = 0$.

If range $f = \{0\}$, the image of everything is zero, so $f = 0$.

Otherwise — we have $\ker f = \{0\}$ and range $f = V'$.

So f is 1-1 and onto.

Now we must check our claim:

proof of claim:

(1) $\ker f$ is a subrepresentation: ie)

Need: $v \in \ker f \Rightarrow \rho(g) v \in \ker f$

or $f(v) = 0$. $\Rightarrow f \rho(g) v = 0$

Now we use the fact that f is an intertwiner.

↳ That is, $f \rho(g) v = \rho(g) f(v) = 0$ when $f(v) = 0$.

(2) range f is a subrep of V'.

Need: $v' \in$ range $f \Rightarrow \rho(g) v' \in$ range f

so $v' = f(v)$ for some $v \in V$ $\Rightarrow \rho(g) v' = f(w)$ for some $w \in V$

But: $\rho(g) v' = \rho(g) f(v) \underset{\text{f is an intertwiner}}{=} f \rho(g) v$

So we use $w = \rho(g) v$. ☐ end proof of claim & lemma

Defn: If $\rho: G \longrightarrow End(V)$ and $\rho': G \longrightarrow End(V')$ are reps and $f: V \longrightarrow V'$ is an intertwiner that's 1-1 and onto, we call it an <u>equivalence</u> (or isomorphism) and we say the reps are <u>equivalent</u>.

Moral: Equivalent reps are "the same" for all practical purposes.

Schur's Lemma (Part 2) — If $\rho: G \longrightarrow End(V)$ is an irred. rep and $f: V \longrightarrow V$ is an intertwiner (then by part 1 f is either zero or 1-1 & onto) then in fact $f = \lambda 1_V$ (a multiple of the identity) for some $\lambda \in \mathbb{C}$.

(*)

proof: Any $f: V \longrightarrow V$ can be written as

$$\left(\begin{array}{c|c|c|c} \begin{matrix}\lambda_1 & 1 & & 0\\ & \ddots & \ddots & \\ & & \ddots & 1\\ 0 & & & \lambda_1\end{matrix} & 0 & 0 & 0 \\ \hline 0 & \begin{matrix}\lambda_2 & 1 & & 0\\ & \ddots & \ddots & \\ 0 & & & \lambda_2\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \ddots & 0 \\ \hline 0 & 0 & 0 & \begin{matrix}\lambda_n & 1 & & 0\\ & \ddots & \ddots & 1\\ 0 & & & \lambda_n\end{matrix} \end{array}\right)$$

T_1 T_n

Jordan canonical form.

in some basis.

we want: $\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ · T_i is a $k_i \times k_i$ matrix

Incorrect! here d_1 below

Let $T = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ a $k \times k$ matrix

then $(T - \lambda I)^k = 0$ $\quad (T - \lambda I) = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & & 0 \end{pmatrix}$

$(T - \lambda I)^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & \ddots & 1 \\ & & \ddots & 0 \\ 0 & & & 0 \end{pmatrix}$ raise to more powers, 1's move up.

If f has blocks $T_1, \dots, T_n$

the char. eqn! $(f-\lambda_1 I)^{k_1}(f-\lambda_2 I)^{k_2} \dots (f-\lambda_n I)^{k_n} = 0$

<u>Claim</u>: Range $(f-\lambda I)^{k_1} \dots \cancel{(f-\lambda_i I)^{k_i}} \dots (f-\lambda_n I)^{k_n}$

(cross of one of them) is a subrep of V.

(f is an intertwiner, so is f^2 since the comp. of intertwiners is again an intertwiner.)

We saw in Schur (Part I) that the range of an intertwiner is a subrep; f is an intertwiner so any poly in f is also.

This range ~~looks like~~: is contained in:

$$\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \hline 0 \\ 0 \\ 0 \\ \hline a \\ b \\ c \\ \vdots \\ \hline 0 \\ 0 \\ 0 \end{pmatrix} \right\} \quad i^{\text{th}} \text{ block}$$

this is supposed to be a sub-rep of V, but V is irreducible!

So, since V is irred, this subrep must be either

$\{0\}$ or V.

The only way this

proof:

① alg. closed, so f has an eigenvalue, λ.
Consider
$$(f-\lambda I).$$
This has nontrivial kernel : $\ker(f-\lambda I) \neq 0$

$\Rightarrow \ker(f-\lambda I) = V$ since V is irreducible
(only subreps are $\{0\}$ and V
and $\neq 0\}$.

Thus $f = \lambda I$.

$\square$