4/3/01

Modify our spin network technology slightly to make there be only finitely many vector spaces "j". Before, we started w/

$V = \mathbb{C}^2$ and defined symmetrizers

$p_S : V^{\otimes n} \longrightarrow V^{\otimes n}$ by

$p_S : \frac{1}{n!} \sum_{\sigma \in S_n} \boxed{\sigma}$ ← n strands coming in & going out

and "j" was the vector space which is the range of p_S when $n = 2j$

$n = 0, 1, \ldots$
$j = 0, \frac{1}{2}, \ldots$

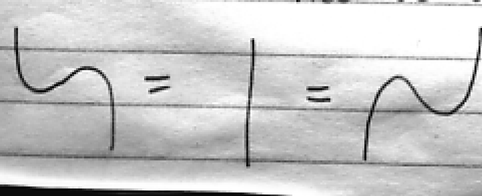
We need to modify this somehow.

Before – everything came from $V = \mathbb{C}^2$ together w/ a symplectic structure

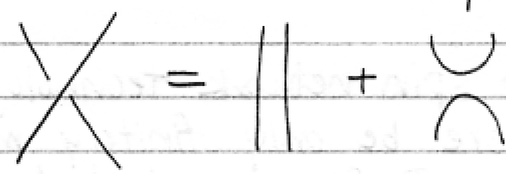
$\omega : V \otimes V \longrightarrow \mathbb{C}$

which we drew as:  cup

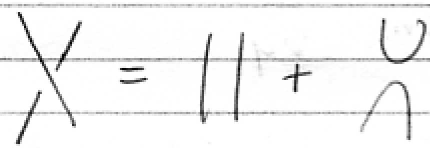
This "∪" determines a unique "∩" cap st

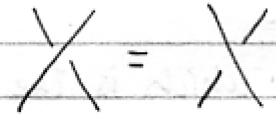


Then – we have a binor identity



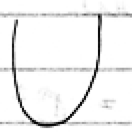
<u>Note:</u> $\Downarrow$ rotating above eqn. we get

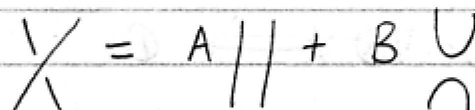


$\Rightarrow$  so "braiding is a symmetry"
(over/under crossings don't matter)

We'll modify the binor identity.

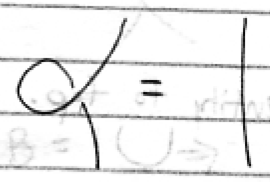
Let's start w/ $V = \mathbb{C}^2$ and an operator

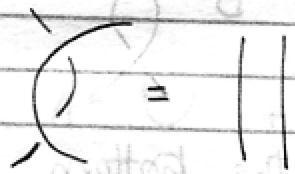
 $V \otimes V \to \mathbb{C}$

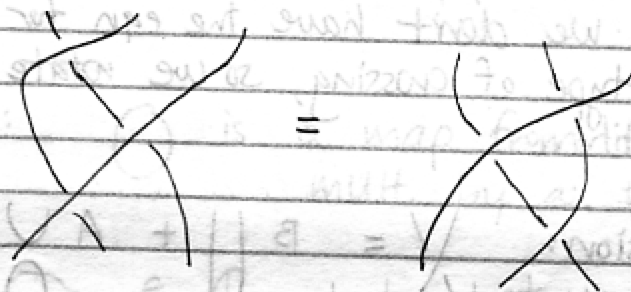
satisfying  where $A, B \in \mathbb{C}$.

We'd like to pick A and B st the Reidemeister moves are preserved.

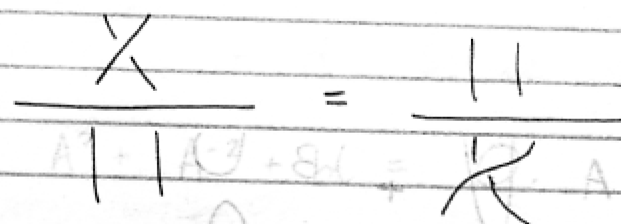
R. moves:

(1) 

(2) 

(3) 

or



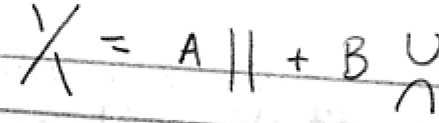
Note: The 1st Reidemeister move didn't hold before!

In fact — when ∪ was a symplectic structure, we had

, so

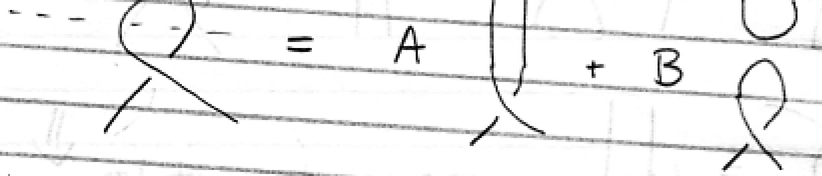
version of R. move #1

The 1st move doesn't hold "up to a phase": 

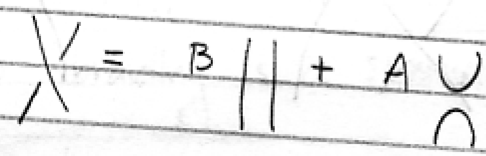
Given 

let's see when R. II move holds.

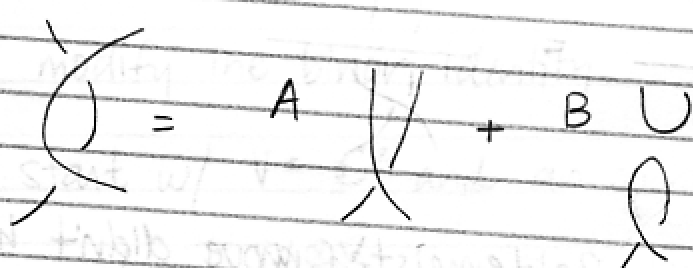
Apply above binor identity to top.

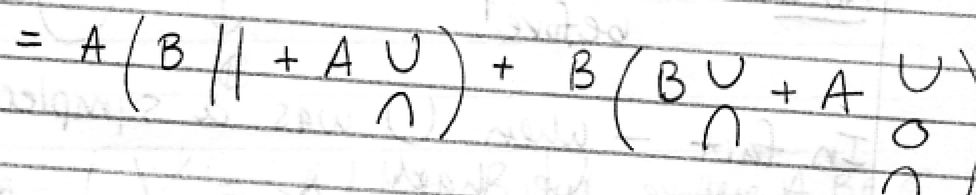


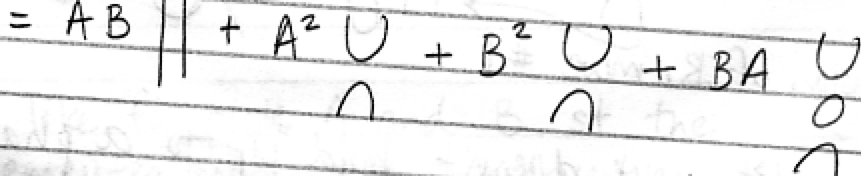
Now we need to deal w/ the bottom crossing. We don't have the eqn for the bottom type of crossing, so we rotate the given identity.

rotated version: 

So:







To get  $= \, ||$ we can assume:

$AB = 1 \Rightarrow B = A^{-1}$ (need $A \neq 0$)

and

the loop from middle (not zero)

$$A^2 \asymp + A^{-2} \asymp + \bigcirc \asymp = \bigcirc$$

<u>Recall</u>: $\bigcirc$ is a map from $\mathbb{C} \to \mathbb{C}$ so it's mult. by a #.

So: $\bigcirc \in \mathbb{C}$, but what # is it?

We'll call it d. So we want:

$$A^2 + A^{-2} + d = 0.$$

This tells us $d = -(A^2 + A^{-2})$

<u>Note</u>: w/ usual binor identity:  $= \, || + \asymp \Rightarrow A = B = 1.$

$\Rightarrow d = -2.$

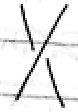
(recall – closed loop evaluated to $-2 = -\dim V$)

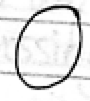
ie) when $A = B = 1$, this gave $\bigcirc = -2$

Recall – our base field is always $\mathbb{C}$.

So – we have one free variable to work with – A (which tells us what B is).

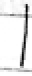
So – we get 2 move if

(*)
$$\left\{ \begin{array}{l} \text{} \end{array} \right.$$
 $=$ A  $+$ A^{-1} 

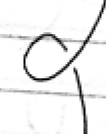
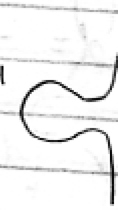
 $= -(A^2 + A^{-2})$

This technique/eqns is called "Kauffman skein relations" & discovered by Louis Kauffman. ~1985.

What about the Reidemeister I move?

Is  $=$  ?

by above (*)

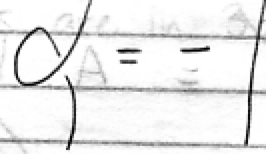
 $= A$   $+ A^{-1}$ 

$= \left[-A\left(A^2 + A^{-2}\right) + A^{-1}\right]$ 

$= \left[-A^3 - A^{-1} + A^{-1}\right]$ 

$= -A^3$ 

We had $A=1 \Rightarrow$

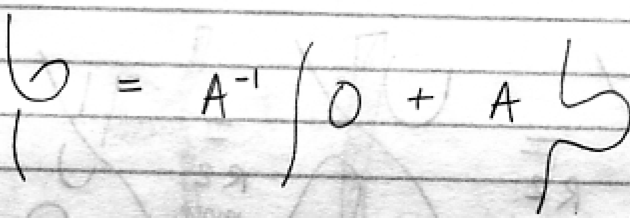


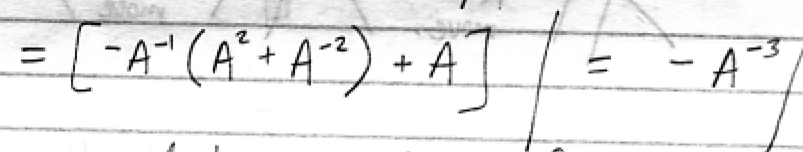
(rotating an electron 360°, = − old vector)
this is a fermion.
(bozon − stays same (sign))

"Anyons" : A can be "anything" (like a bozon/fermion)

---

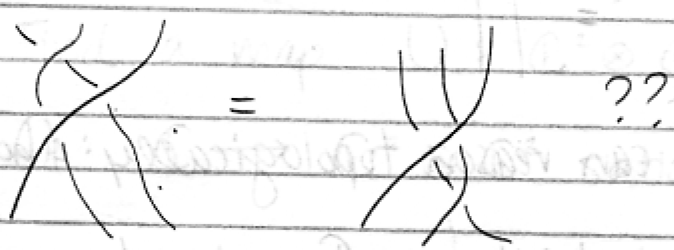
the reverse 360° rotation



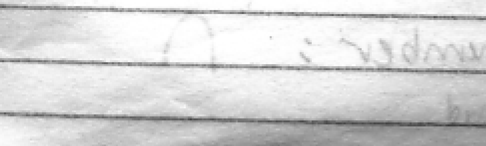


(the inverse of the complex # from before)

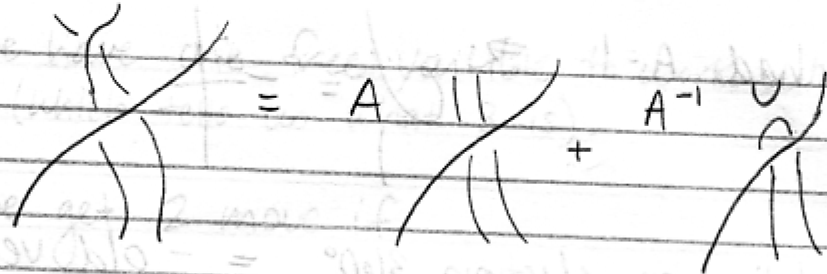
What about Reidemeister III move? Is



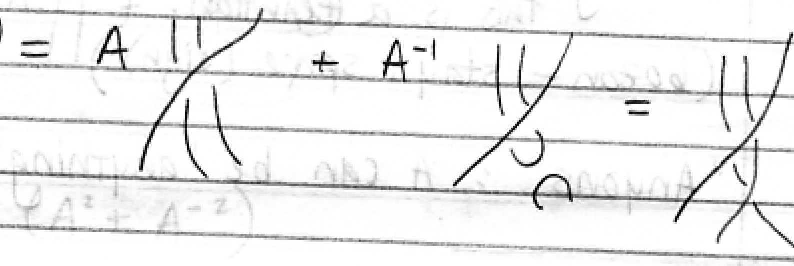
We can use brute force to change each crossing: using



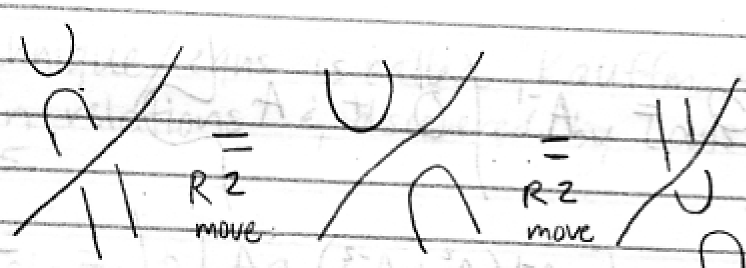
proof:



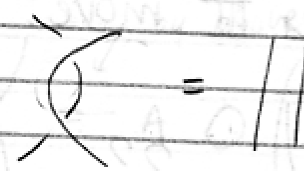
by 2 applications of R. 2 move! (*)



(*)



Recall— R2 move lets us slide lines underneath one another.



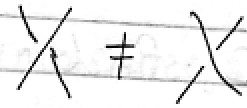
So— we can reason topologically about

$\cup : V \otimes V \longrightarrow \mathbb{C}$ and

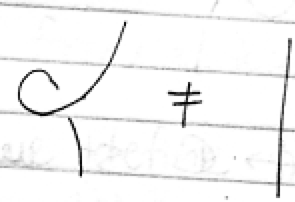
$\cap : \mathbb{C} \longrightarrow V \otimes V$ if we

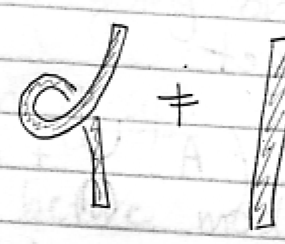
remember:

① Now the pictures are in 3d, not 4d.

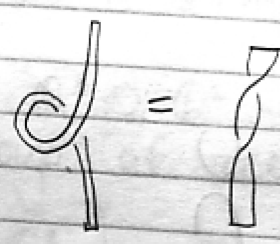


② The strands are really ribbons (as before)





but



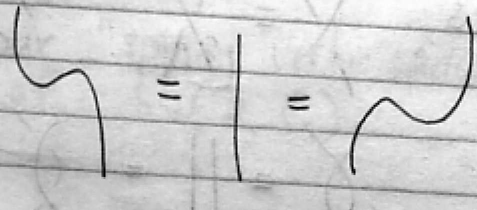
---

this map determined by 4 #'s a,b,c,d

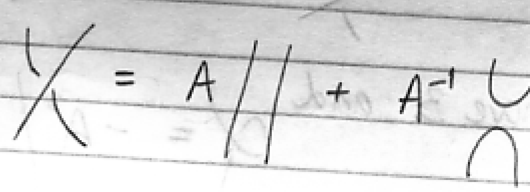
HW: Find a map $\cup : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}$ st

if $\cap : \mathbb{C} \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$ satisfies

Find $\cap$ st these hold (uniquely determined)



then we also have

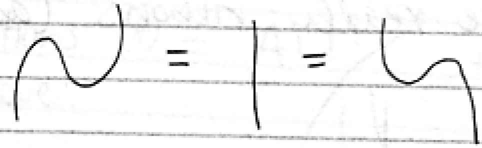


<u>Note</u>: this won't work if it's usual braid/switch map!

HW: Choose $A \neq 0$ in $\mathbb{C}$.

① Given $\cup : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}$, find

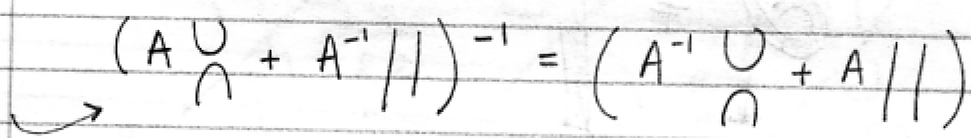
$\cap : \mathbb{C} \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$ st



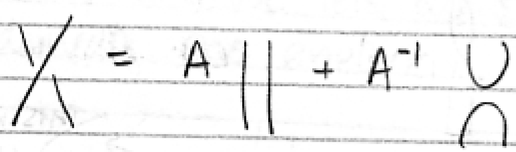
② Find $\cup : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}$ st w/ $\cap$

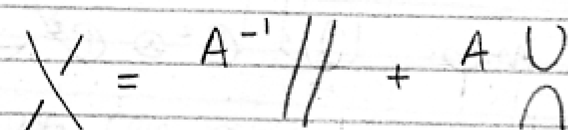
defined as above, we get

4x4 inverse of matrix

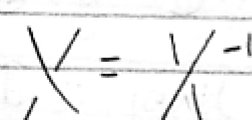


Then – you can set

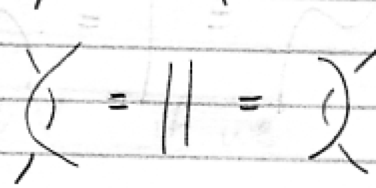




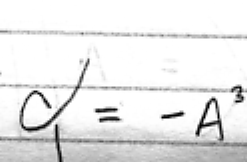
and ② guarantees



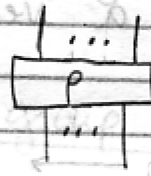
ie)



So we get R. move 3 and



Next step: To define

"q-symmetrizers" 

using $\times$, $\times$, $\cup$, $\cap$ and we want $p^2 = p$.

The old formula won't work.

Before we define these, let's see what's really going on:

<u>Quantum groups</u>

Recall — before modifying the binor identity, we had a symplectic structure

$\omega: \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}$. We saw that the group of all linear operators

$f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ preserving ω was:

$GL(2, \mathbb{C}) = \{ f: \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \mid \det f = 1 \}$
(all 2x2 matrices (lin transf) w/ det. 1)

All our "spin-j vector spaces" were representations of $SL(2, \mathbb{C})$.

Defn: If G is a group, a <u>representation</u> of G on a vector space V is a map

$$\rho : G \longrightarrow \mathrm{End}(V)$$ — all lin. transf. of V.

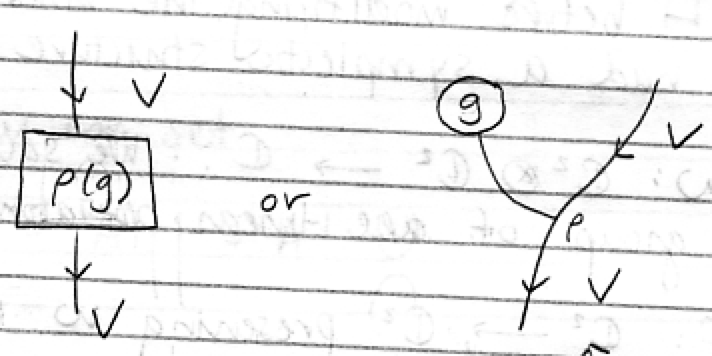
ρ is a homomorphism st

$$\rho(gg') = \rho(g)\rho(g')$$

$$\rho(1) = 1_V$$

representation takes any elt. of G and turns it into a lin. trans. of V.

We can draw $\rho(g) : V \longrightarrow V$ as



we're kinda mixing groups & v. spaces → make all v. spaces by trick on next pg

What's not entirely correct about this?

before we were in the category of v. spaces, but here we have a group elt. g in a picture w/ v. spaces.

It's a bit weird since G isn't a vector space.

But - we can turn a group into a v. space!

We can turn G into a vector space by forming

$\mathbb{C}[G]$ "the group algebra of G"

$= \{$formal lin. combs of elts of $G\}$

A typical elt looks like: $\sum_{g \in G} a_g g \in \mathbb{C}[G]$, $a_g \in \mathbb{C}$

$\mathbb{C}[G]$ is an algebra w/ mult. defined as:

$$\left(\sum_{g \in G} a_g g\right) * \left(\sum_{h \in G} b_h h\right) = \sum_{g,h \in G} a_g b_h gh$$

mult. in the group algebra

So - now g is an elt of the group algebra, $\mathbb{C}[G]$, making our picture on prev. page ok.

Now - given a representation ρ of G on V, we get

$$\rho : \mathbb{C}[G] \longrightarrow \mathrm{End}(V) \text{ by}$$

$$\rho\left(\sum_{g \in G} a_g g\right) = \sum_{g \in G} a_g \rho(g)$$

Note - ρ is linear from $\mathbb{C}[G]$ to $\mathrm{End}(V)$.

We also have $\rho : \mathbb{C}[G] \longrightarrow V \otimes V^*$ or

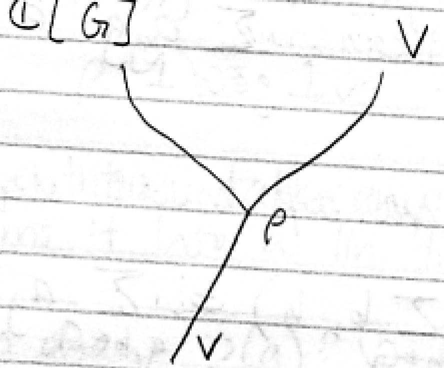
$\rho : \mathbb{C}[G] \otimes V \longrightarrow V$ (equivalent)

$$\rho : \mathbb{C}[G] \longrightarrow V \otimes V^*$$

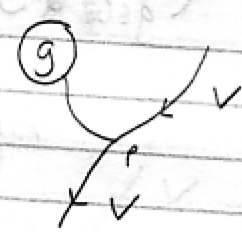
or

$$\rho : \mathbb{C}[G] \otimes V \longrightarrow V \qquad \rho\left(\sum a_g g \otimes v\right) = \sum a_g \rho(g) v$$

We draw this as:



or if $g \in G \subseteq \mathbb{C}[G]$, we get

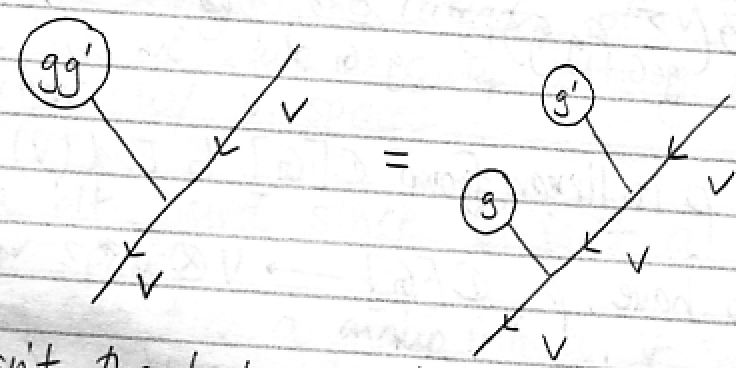


Now — ρ is a representation if

① $\rho(gg') = \rho(g)\rho(g')$ &

② $\rho(1) = 1_V$

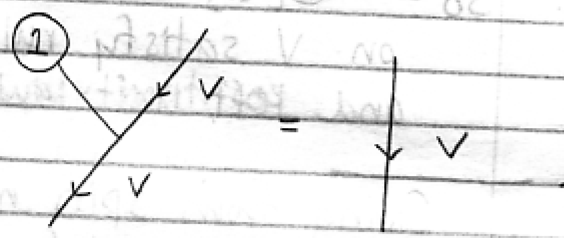
①



This isn't the best way to draw this...

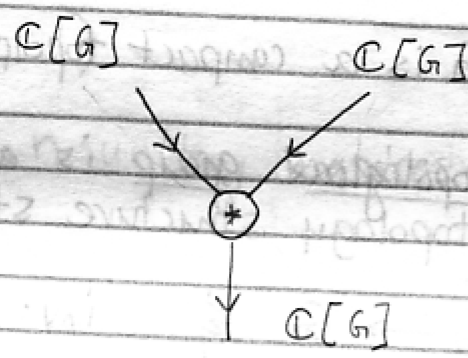
② $\rho(\underline{1}) = 1_V$

(resembles left unit law)

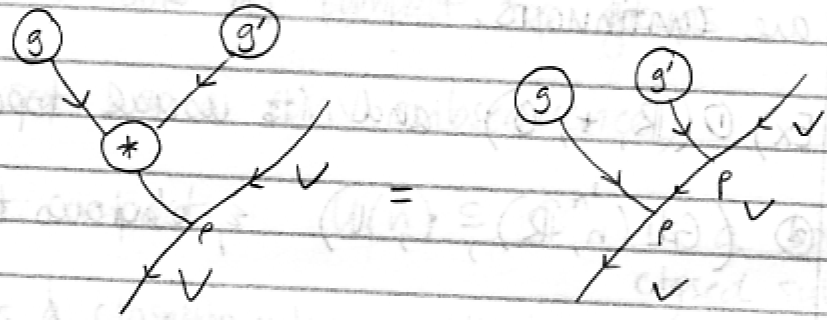


① looks better if we <u>draw</u> multiplication gg' in G (or $\mathbb{C}[G]$) using pictures.

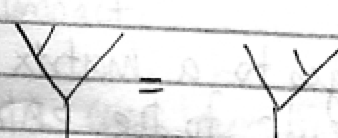
$$* : \mathbb{C}[G] \otimes \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$$



This gives



This is like the assoc. law:



So — $\mathbb{C}[G]$ and its representation ρ on V satisfy rules just like assoc. and left unit law.

Going from spin networks to our new "q-deformed spin networks" is really replacing $\mathbb{C}[SL(2,\mathbb{C})]$ by a different algebra — "quantum $SL(2,\mathbb{C})$" — which depends on A.

(when $A=1$, we have old $SL(2,\mathbb{C})$.)

Track 2

Suppose G is a compact topological group.

Defn: A topological group is a group G w/ a topology structure st

$$m: G\times G \to G \qquad \text{\&} \qquad inv: G \to G$$
$$(g,h) \mapsto gh \qquad\qquad g \mapsto g^{-1}$$

are continuous.

Ex) ① $(\mathbb{R}, +, 0)$ and its usual topology.

② $(GL(n,\mathbb{R}), \cdot, 1)$ & obvious topology

nxn invertible real matrices

w/ matrix mult.

not compact, 1 (id matrix), 2 (id) 3 (id) ... sequence w/ no subsequence

(seq. of matrices converges to a matrix if the entries converge to the entries in the limiting matrix)

③ $(GL(n,\mathbb{C}), \cdot, 1)$

nxn inv. complex matrices

④ $(SL(n,\mathbb{C}), \cdot, 1)$

nxn complex complex matrices w/ det = 1.

⑤ $(SU(n), \cdot, 1)$

nxn unitary matrices w/ det 1.

⑥ $(U(n), \cdot, 1)$

nxn unitary matrices meaning

$UU^* = 1$

$\Rightarrow U^* = U^{-1}$

conjugate transpose

We want to look at <u>compact</u> top. groups (every seq. has a conv. subsequence)

ex) ⑥ is compact. $U(n) \subseteq \mathbb{C}^{n^2}$ is a bounded & closed subset hence, compact.

so is ⑤ since this a closed subset of ⑥ hence compact.

det is a cont funct. of the matrix entries.

set of values of cont funct = constant is a closed subset!

preimage under cont funct. of a single point is closed

(inv. image of closed set is closed under a cont funct)

ex) $SU(2)$ compact top. group.

<u>Thm</u>: $\exists$ unique Borel measure, dg, on any compact topological group s.t.

① $\int_G 1\, dg = 1$ (normalization)

② $\int_G f(g)\, dg = \int_G f(hg)\, dg \quad \forall h \in G$

left translation/ mult

similar: $\int_{\mathbb{R}} f(x)\, dx = \int_{\mathbb{R}} f(x+h)\, dx$

the integral is translation invariant.
(specifically - left translation invariance)

③ $\int_G f(g)\, dg = \int_G f(gh)\, dg \quad \forall h \in G$

right translation invariant.

④ $\int_G f(g)\, dg = \int_G f(g^{-1})\, dg$

invasion invariance

for any integrable funct. $f: G \longrightarrow \mathbb{R}$.

This measure is called <u>normalized Haar measure</u>. Given this, we can define $L^2(G)$, where

$$L^2(G) = \{ f : G \to \mathbb{C} \mid \int_G |f|^2 dg < \infty \}$$

is a Hilbert space.

<u>Goal</u>: get a nice orthonormal basis of $L^2(G)$. The Peter-Weyl Thm gives us this basis.

$L^2(G)$ is a Hilbert space & algebra, not necess. finite dim'l, not necess. w/ mult. unit 1). If G is finite — we're okay.

The product in $L^2(G)$ is called <u>convolution</u> and goes like this:

If $\psi, \phi \in L^2(G)$ then

$$(\psi * \phi)(g) = \int_G \psi(gh)\, \phi(h^{-1})\, \underbrace{dh}_{\text{Haar measure}}$$

Why is $\psi * \phi \in L^2(G)$?

Show $\psi * \phi \in L^2(G)$, we know $\psi, \phi \in L^2(G)$.

$\psi \in L^2(G)$ iff $\|\psi\| < \infty$ where

$$\|\psi\|^2 = \int_G |\psi(g)|^2 \, dg$$

Show $\|\psi * \phi\|^2 < \infty$.

$$\|\psi * \phi\|^2 = \int_G |\psi * \phi(g)|^2 \, dg$$

$$= \int_G \left| \int_G \psi(gh)\, \phi(h^{-1})\, dh \right|^2 dg$$

$$\leq \int_G \left(\int_G |\psi(gh)\, \phi(h^{-1})| \, dh \right)^2 dg$$

use Cauchy-Schwarz inequality:

$$\int \psi \phi \, dx \leq \underbrace{\left(\int |\psi|^2 dx \right)^{1/2} \left(\int |\phi|^2 dx \right)^{1/2}}_{\|\psi\| \, \|\phi\|}$$

$$\leq \int_G \|\psi(g\,\cdot)\|^2 \, \|\phi(\cdot^{-1})\|^2 \, dg$$

Using Haar's thm — these are constants now! no longer functs. (not depend on g)

$$= \int_G \|\psi\|^2 \|\phi\|^2 \, dg$$

$$= \|\psi\|^2 \|\phi\|^2 \underbrace{\int_G 1 \, dg}_{= 1}$$

1 since we've got compactness.

which is finite since $\psi, \phi \in L^2(G)$.

<u>Note</u>: $*$ is associative : proof is straightforward.

Ex) G is a finite group. Then $L^2(G) \cong \mathbb{C}[G]$.

ie) $L^2(G) \cong \mathbb{C}[G]$

$$\psi \longmapsto \sum \psi(g) g$$

$$\psi, \text{st } \psi(g) = a_g \longleftarrow \sum a_g g$$

Then — δ_g are a basis of $L^2(G)$ where

$$\delta_g(h) = \begin{cases} 1 & g = h \\ 0 & \text{otherwise} \end{cases}$$

They're orthogonal but not normalized.

$$\|\delta_g\|^2 = \int_G |\delta_g(h)|^2 \, dh$$

$$= \sum_{h\in G} |\delta_g(h)|^2 \frac{1}{|G|}$$

order of group

the term $\frac{1}{|G|}$ needed for $\int_G 1 \, dg = 1$.

$$= \frac{1}{|G|} .$$

So, we get an orthonormal basis of $L^2(G)$ by using

$$\psi_g = \sqrt{|G|}\, \delta_g$$

* But this trick only works if G is finite. *

We can work out $\delta_g * \delta_h = \underline{\quad\quad} \delta_{gh}$

fudge factor

or better:

$$\psi_g * \psi_h = \psi_{gh}$$

Note: the mult. in Hilbert space $L^2(G)$ comes from product in the group.

Goal: Do something like this for G infinite

ex) $SU(2)$:

we want a nice orthonormal basis of $L^2(G)$ and nice formula for $*$ in terms of this basis.