(*) If $K = e^{\hbar H}$, $K \otimes K = e^{\hbar H} \otimes e^{\hbar H}$

$$= e^{\hbar (H \otimes H)}$$

$$K^2 \otimes K^2 = e^{2\hbar(H \otimes H)}$$

5/1/01

let $\boxed{q = e^{\hbar}}$ and let $U_q\, sl(2, \mathbb{C})$ be the algebra generated by E, F, K w/ relations:

$$\begin{cases} EF - FE = \dfrac{K^2 - K^{-2}}{q - q^{-1}} \\ KE = q EK \\ KF = q^{-1} FK \end{cases}$$

It becomes a bialgebra w/

$$\Delta E = E \otimes K + K^{-1} \otimes E$$

$$\Delta F = F \otimes K^{-1} + K \otimes F$$

$$\Delta K = K \otimes K$$

e the co-unit, one of the maps we defined before.

$$e(E) = e(F) = 0, \quad e(K) = 1$$

In fact, $U_q\, sl(2, \mathbb{C})$ is quasitriangular with

$$R \in U_q sl(2, \mathbb{C}) \otimes U_q sl(2, \mathbb{C})$$

where

$$R = \sum_{n=0}^{\infty} \frac{q^{\frac{1}{2}n(n+1)} (1 - q^{-2})^n}{[n]_q!} e^{2\hbar(H \otimes H)} E^n \otimes F^n$$

(*)

Here, the "q-integer" $[n]_q$ is

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Note — as $q \to 1$ then $[n]_q \to n$.

$$\frac{e^{n\hbar} - e^{-n\hbar}}{e^{\hbar} - e^{-\hbar}} \longrightarrow \frac{(1+n\hbar) - (1-n\hbar)}{\hbar + \hbar} = n.$$

Also — the "q-fractional" $[n]_q!$ is given by

$[n]_q \cdots [1]_q$ and $[0]_q! = 1$.

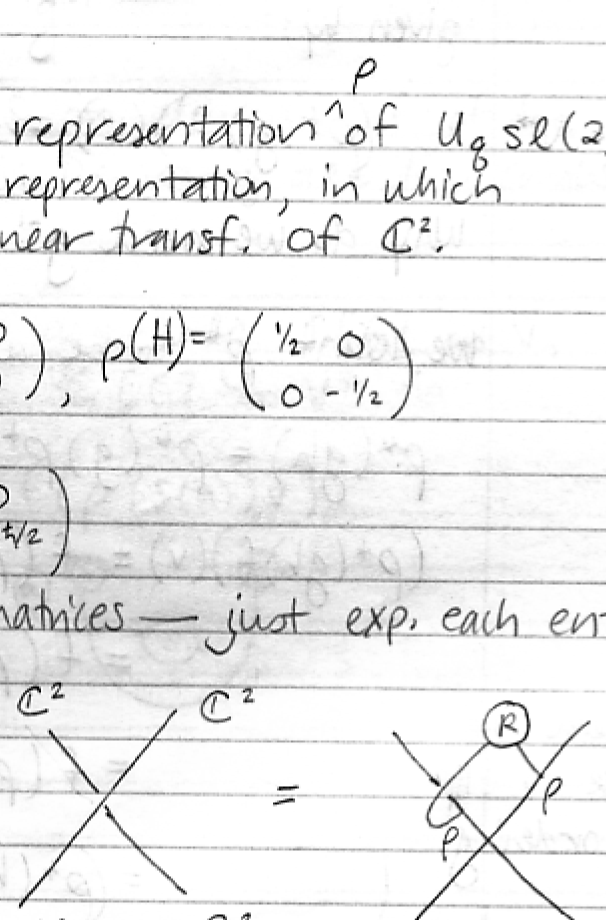
$\boxed{\text{HW\#5}}$: There is a 2-dim'l representation ρ of $U_q\, sl(2,\mathbb{C})$ called the "spin-$\frac{1}{2}$" representation, in which E, F, H act as these linear transf. of $\mathbb{C}^2$.

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ \rho(H) = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$$

and $\rho(K) = \rho(e^{\hbar H}) = \begin{pmatrix} e^{\hbar/2} & 0 \\ 0 & e^{-\hbar/2} \end{pmatrix}$

exponentiating diagonal matrices — just exp. each entry.

Given this, work out



explicitly in terms of e_1, e_2 basis of $\mathbb{C}^2$.

This should be same operator

$B : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$ that we wrote down in class, satisfying $\times = A \,||\, + A^{-1} \genfrac{}{}{0pt}{}{\cup}{\cap}$.

Note – as soon as $n=2$, $E^2 = 0$, $F^2 = 0$
so only 2 terms to look at: $n=0$, $n=1$.

<u>Note</u> – once we know what our rep. ρ does to the basis elements, we know what it does to all the elements in the algebra.

Given a group G and a rep. $\rho : G \to \mathrm{End}(V)$ there's a <u>dual rep</u> $\rho^* : G \longrightarrow \mathrm{End}(V^*)$ given by:

$f \in V^*$
$$(\rho^*(g) f)(v) = f(\rho(g^{-1}) v)$$

Why do we need g^{-1} above?

We want ρ^* to be a rep, so check

$$\rho^*(gh) = \rho^*(g) \rho^*(h)$$

$$\begin{aligned}
(\rho^*(gh) f)(v) &= f(\rho(gh)^{-1} v) \\
&= f(\rho(h^{-1} g^{-1}) v) \quad \text{since } \rho \text{ is a rep} \\
&= f(\rho(h^{-1}) \rho(g^{-1}) v) \\
&= (\rho^*(h) f)(\rho(g^{-1}) v) \\
&= (\rho^*(g) \rho^*(h) f)(v).
\end{aligned}$$

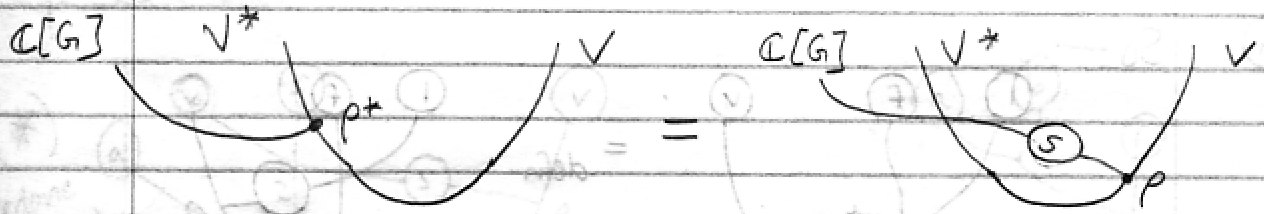
* we need the inverse to flip the ordering here.

Recall: reps for groups in 1-1 correspondence w/ reps for group algs.

we can't talk about dual reps for reps on algebras or bialgebras.

To get a bialgebra to have dual reps, it needs some extra structure — it should be a Hopf algebra.

Let's draw $(\rho^*(g)f)(v) = f(\rho(g^{-1})v) \quad f \in V^*$

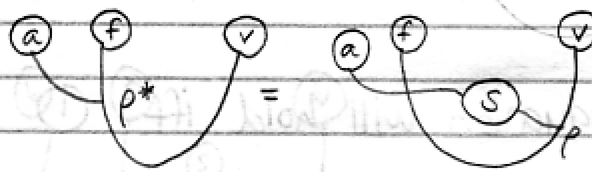


Here — $S: \mathbb{C}[G] \longrightarrow \mathbb{C}[G]$ is:
$$g \longmapsto g^{-1}$$

We need only to define S on elements of of the group G, since $\mathbb{C}[G]$ is the set of all lin. combs of elts of G.

Given a rep ρ of the group alg. $\mathbb{C}[G]$ on V, we get a rep. ρ^* of $\mathbb{C}[G]$ on V^* via:

$$(\rho^*(a)f)(v) = f(\rho(S(a))v)$$

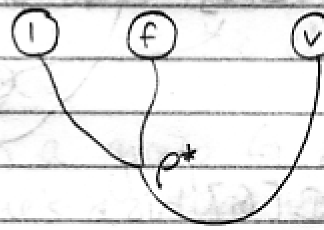


Given a bialg A and linear operator $S: A \longrightarrow A$, and defining ρ^* in terms of ρ, using this formula, when is ρ^* going to be a rep? (assuming ρ is a rep).

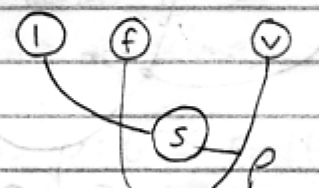
<u>Need</u>: Ⓐ $\rho^*(1) = 1_{V^*}$

Ⓑ $\rho^*(ab) = \rho^*(a)\rho^*(b)$

Ⓐ So —



$=_{\text{defn}}$



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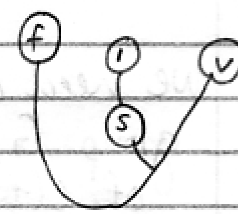

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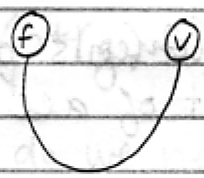
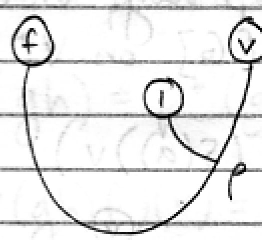


true $\forall f, v$ iff $\rho^*(1) = 1$.

but —



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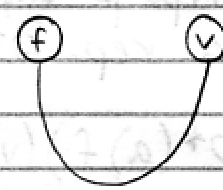
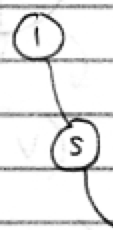


So, our equation will hold iff

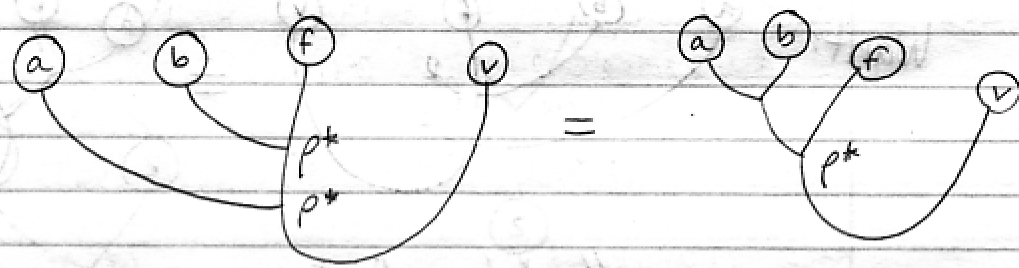


$=$

or $S(1) = 1$.

⑧ Check $\rho^*(ab) = \rho^*(a)\rho^*(b)$

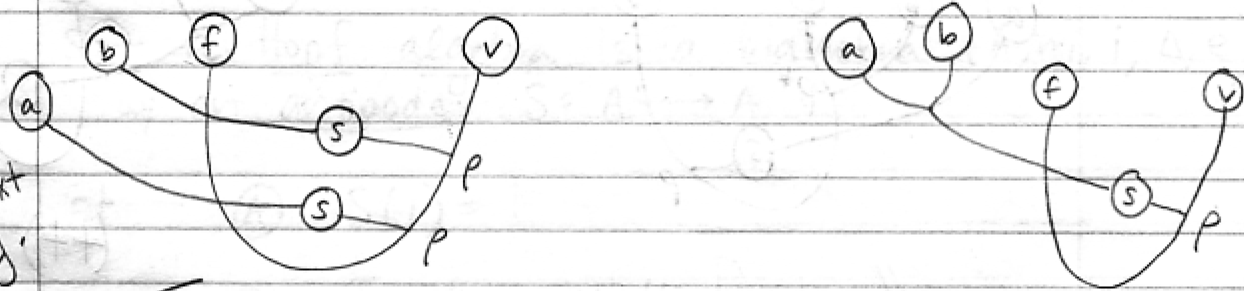
Want



by defn of ρ^* used twice //

// defn.

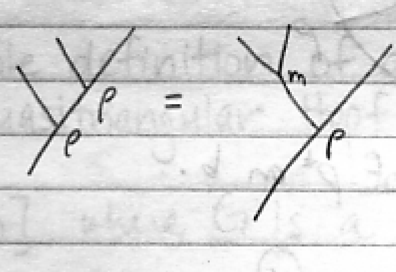
(*) redone on next pg.



what property of S makes this work?

* ρ has the property that $\rho(ab) = \rho(a)\rho(b)$

ie)



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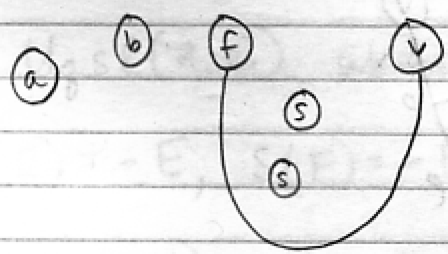
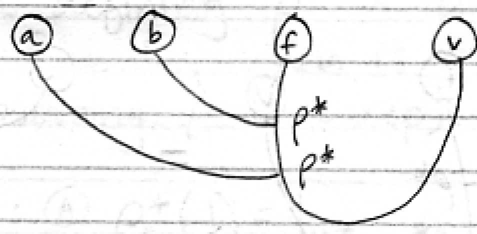


Ⓑ Check if $\rho^*(ab) = \rho^*(a)\rho^*(b)$

Want:



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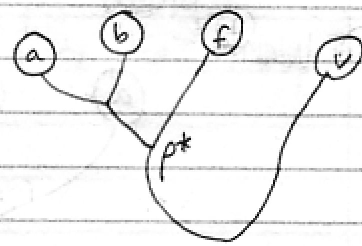
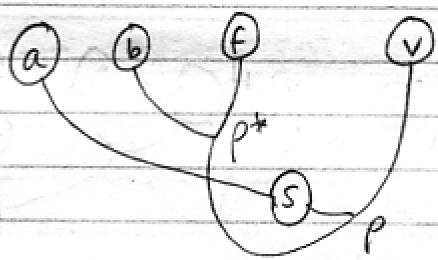
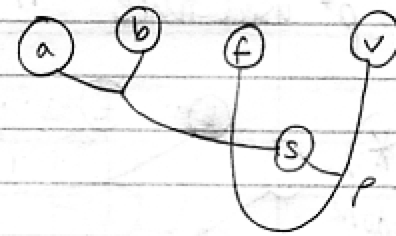


$\parallel$ defn of ρ^* on a

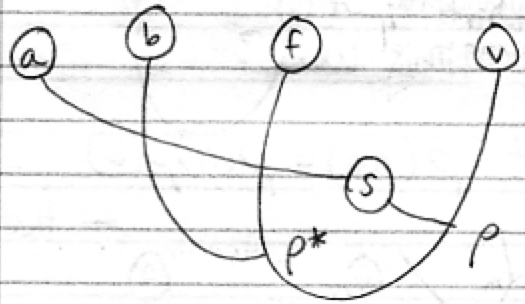


$\parallel$

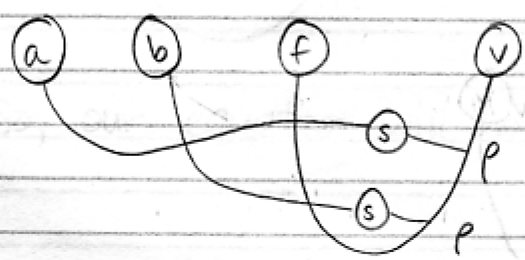


(‡)

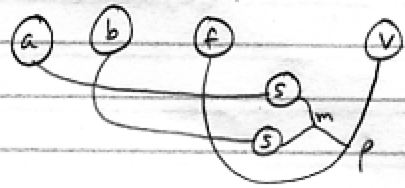
$\parallel$ by topology



$\parallel$ defn of ρ^* on b.

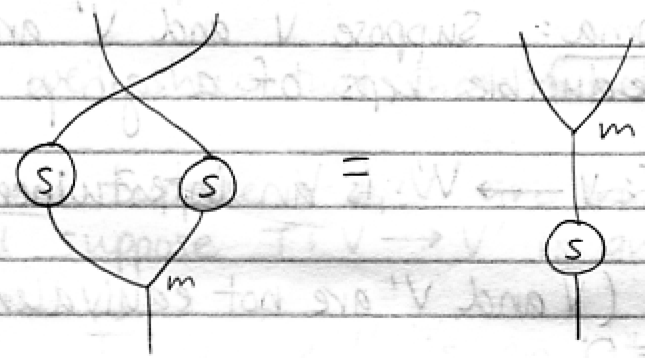


$\parallel$



(†)

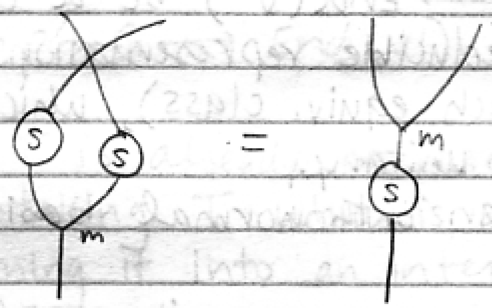
So, (t) = (t t) will hold if



So — a <u>Hopf algebra</u> is a bialgebra (A, m, i, Δ, e) w/ an <u>antipode</u> $S: A \longrightarrow A$

st (A) $S(1) = 1$

(B)



<u>Defn</u>: A possible definition of a "<u>quantum group</u>" is a quasitriangular Hopf algebra.

Exs) $\mathbb{C}[G]$ where G is a group.

$U_q sl(2; \mathbb{C})$ where

$S(E) = -E$, $S(F) = -F$, $S(K) = K^{-1}$

ie) $S(H) = -H$ since $K = e^{\hbar H}$

<u>Track 2</u>: G is a compact group.

<u>Schur's lemma</u>: Suppose V and V' are two irreducible reps of a group G.
Suppose
$$T : V \longrightarrow V' \text{ is an intertwiner.}$$

If $V \not\cong V'$ (V and V' are not equivalent), then $T = 0$.

If $V = V'$, then $T = cI$ for some $c \in \mathbb{C}$ (some multiple of the identity).

<u>Peter-Weyl Thm</u>: If G is a compact group, let $\rho^\alpha : G \longrightarrow \mathrm{End}(V^\alpha)$ be a complete set of irreducible representations (one from each equiv. class), which we can assume are unitary.
Let e_i^α be an orthonormal basis of V^α.

In this basis, we can write $\rho^\alpha(g) : V^\alpha \to V^\alpha$ as a matrix:

$$\rho_{ij}^\alpha(g) = \langle e_j, \rho^\alpha(g) e_i \rangle, \quad 1 \le i,j \le \dim V^\alpha$$

Then, the functs
$$\sqrt{\dim V^\alpha}\, \rho_{ij}^\alpha$$
form an o.n. basis of $L^2(G)$.

proof: We must show they are orthonormal & that they are a basis (ie. span in particular)

First we'll show that these functions are orthonormal.

Lemma: Suppose V & V' are irred. reps of G and suppose $T: V \longrightarrow V'$ is any linear operator. Then

$\tilde{T}: V \longrightarrow V'$ is an intertwiner where

$$\tilde{T}(v) = \int \rho'(g)\, T \rho(g^{-1}) v \; dg$$

and where

$$\left.\begin{array}{l} \rho: G \longrightarrow \mathrm{End}(V) \\ \rho': G \longrightarrow \mathrm{End}(V') \end{array}\right\} \text{ different reps.}$$

Note, if T is already an intertwiner, then $\tilde{T} = T$. We are taking T which is any linear operator and turning it into an intertwiner $\tilde{T}$.

So, $\tilde{T} = 0$ if $V \not\cong V'$ and $\tilde{T} = cI$ if $V = V'$ by Schur's lemma.

pf of lemma: We need $\tilde{T}\rho(h) = \rho'(h)\tilde{T}$ $\forall h \in G$.

$$\tilde{T}\rho(h)v = \int \rho'(g)\, T\rho(g^{-1})\rho(h) v \; dg$$

since ρ is a rep.

$$= \int \rho'(g)\, T\rho(g^{-1}h) v \; dg$$

Let $k^{-1} = g^{-1}h$.

let $g^{-1}h = K^{-1}$.

Now, by properties of Haar measure:

$$dK = dg$$

So, $K = h^{-1}g$ and $hK = g$

so from prev pg. $= \int \rho'(hK) T \rho(K^{-1})\, dK$

$$= \int \rho'(h) \rho'(K) T \rho(K^{-1})\, dK$$

$$= \rho'(h) \tilde{T}.$$

$\blacksquare$ pf of lemma

<u>Moral of lemma:</u> We can average any linear operator to get an intertwiner.

Given $e_i^\alpha \in V^\alpha$ and $e_j^\beta \in V^\beta$, we get

$$T : V^\alpha \longrightarrow V^\beta$$

by: $T = |e_j^\beta\rangle\langle e_i^\alpha|$

ie)

$$\underset{V}{\underset{\cap}{Tv}} = e_j^\beta \langle e_i^\alpha, v\rangle = \langle e_i^\alpha, v\rangle e_j^\beta$$

multiplication on the right

$T: V^\alpha \longrightarrow V^\beta$

$\rho = \rho^\alpha \qquad \rho^\beta = \rho'$

So by our lemma - we know

$\tilde{T} = 0$ if $\alpha \neq \beta$,
$\tilde{T} = cI$ if $\alpha = \beta$.

If $\alpha \neq \beta$: $\tilde{T} = 0$ iff $\tilde{T} e_k^\alpha = 0 \ \forall K$

iff $\langle e_\ell^\beta, \tilde{T} e_k^\alpha \rangle = 0 \quad \forall \ell.$

where $T = |e_j^\beta\rangle\langle e_i^\alpha|$

so -

$$\langle e_\ell^\beta, \int \rho'(g) |e_j^\beta\rangle\langle e_i^\alpha| \rho(g^{-1}) e_k^\alpha \, dg \rangle = 0$$

$$\Rightarrow \langle e_\ell^\beta, \int \rho'(g) \, e_j^\beta \langle e_i^\alpha, \rho(g^{-1}) e^k \rangle \, dg \rangle = 0$$

inner product is linear,
so we can push it into
the integral

$$\text{so, } (*) \int \langle e_\ell^\beta, \rho'(g) e_j^\beta \rangle \langle e_i^\alpha, \rho(g^{-1}) e_k^\alpha \rangle \, dg = 0$$

where
$\rho = \rho^\alpha$ and $\rho' = \rho^\beta$ are the reps of G on V^α and V^β respectively.

Recall: $\rho_{ij}^\alpha(g) = \langle e_i^\alpha, \rho^\alpha(g) e_j \rangle$

If ρ is unitary, $\rho(g)\rho(g)^* = I$
$\rho(g^{-1}) = \rho(g)^{-1} = \rho(g)^*$

So, by remarks on bottom of last page:

$$\langle e_i^\alpha, \rho(g^{-1}) e_k^\alpha \rangle = \langle e_i^\alpha, \rho(g)^* e_k^\alpha \rangle$$

$$= \langle \rho(g) e_i^\alpha, e_k^\alpha \rangle$$

complex conj. $$= \overline{\langle e_k^\alpha, \rho(g) e_i^\alpha \rangle}$$

$$= \overline{\rho_{ki}^\alpha (g)}$$

Thus, (*) our integral on prev pg. becomes:

$$\int \rho_{\ell j}^\beta(g) \overline{\rho_{ki}^\alpha(g)} \, dg = 0$$

So- if $\alpha \neq \beta$, ρ_{ki}^α and $\rho_{\ell j}^\beta$ are orthonormal!

If $\alpha = \beta$, we know

$$\tilde{T} = c I$$

$T = |e_i^\alpha\rangle\langle e_j^\alpha|$

$$\tilde{T} e_k^\alpha = c e_k^\alpha \quad \forall k$$

$$\langle e_\ell^\alpha, \tilde{T} e_k^\alpha \rangle = c \langle e_\ell^\alpha, e_k^\alpha \rangle = c\, \delta_{\ell k}$$

where e_i^α is an o.n. basis of V^α.

$T = |e_i^\alpha\rangle\langle e_j^\alpha|$

Properties of Haar measure:

$$\int f(g^{-1})\,dg = \int f(g)\,dg = \int f(hg)\,dg = \int f(gh)\,dg$$

$$\int \langle e_\ell^\alpha, \rho^\alpha(g)\, T \rho^\alpha(g^{-1})\, e_k^\alpha\rangle = c\,\delta_{\ell k}$$

or $$\int \langle e_\ell^\alpha, \rho^\alpha(g)\, e_i^\alpha\rangle\langle e_j^\alpha, \rho^\alpha(g^{-1})\, e_k^\alpha\rangle = c\,\delta_{\ell k}$$

Let $e_i = e_i^\alpha$, $\rho = \rho^\alpha$. Then, this says:

$$(\star)\quad \int \rho_{\ell i}(g)\,\overline{\rho_{kj}(g)}\,dg = c\,\delta_{\ell k}$$

This says:

$$\langle \rho_{kj}, \rho_{\ell i}\rangle = c\,\delta_{\ell k}$$

$k=\ell$ or get zero.

this depends on i and j since T did, so call it c_{ij}.

$$\langle \rho_{kj}, \rho_{\ell i}\rangle = c_{ij}\,\delta_{\ell k}$$

Now, substituting $g \to g^{-1}$ in $(\star)$, we get

$$\int \rho_{\ell i}(g^{-1})\,\overline{\rho_{kj}(g^{-1})}\,dg = c_{ij}\,\delta_{\ell k}$$

$$\int \overline{\rho_{i\ell}(g)}\,\rho_{jk}(g)\,dg = c_{ij}\,\delta_{\ell k}$$

$$\boxed{\rho_{ij}(g^{-1}) = \overline{\rho_{ji}(g)} \text{ whenever } \rho \text{ is unitary.}}$$

$$\langle \rho_{i\ell}, \rho_{jk}\rangle = c_{ij}\,\delta_{\ell k}$$

need $\ell = k$ or it's zero.

We want to say something about c_{ij}.

$$\langle \rho_{i\ell}, \rho_{jk} \rangle = c_{ij}\,\delta_{\ell k}$$

need $\ell = k$ or it's zero.

1st one said $\langle \rho_{i\ell}, \rho_{jk} \rangle = c_{k\ell}\,\delta_{ij}$

So — c_{ij} can't depend on the indices at all, unless they aren't equal.
(front indices match, back indices match)

So —

$$\langle \rho_{i\ell}, \rho_{jk} \rangle = c\,\delta_{ij}\,\delta_{\ell k}$$

So —

$\frac{1}{\sqrt{c}}\rho_{ij}$ will be an orthonormal basis. What is c?

$$\rho(g)\rho(g^{-1}) = I$$

T:

$$\Rightarrow \sum_{\ell} \rho_{i\ell}(g)\rho_{\ell k}(g^{-1}) = \delta_{ik}$$

$$\Rightarrow \sum_{\ell} \rho_{i\ell}(g)\overline{\rho_{k\ell}(g)} = \delta_{ik}$$

$$\Rightarrow \sum_{\ell} \int \rho_{i\ell}(g)\overline{\rho_{k\ell}(g)}\,dg = \delta_{ik}$$

Note:
$\langle , \rangle$ is an integral

$$\Rightarrow \sum_{\ell} \langle \rho_{k\ell}, \rho_{i\ell} \rangle = \delta_{ik}$$

$$\Rightarrow \sum_{\ell} c\, \delta_{ki} \underbrace{\delta_{\ell\ell}}_{1} = \delta_{ik}$$

Now if $i = k$, $\sum_{\ell} c = 1$

$$\dim(V^\alpha)\, c = 1$$

$$c = \frac{1}{\dim(V^\alpha)}.$$

So — $\sqrt{\dim V^\alpha}\, \rho_{ij}^{\alpha}$ is orthonormal.

Now — why are they a basis? In particular, how can they both span?

Next — why are they a basis, i.e. why are finite linear combs. of them dense in $L^2(G)$?

We know $C(G)$ is dense in $L^2(G)$. So — need finite linear combs. are dense in $C(G)$. Use Stone-Weierstrass Thm, which says:

If $A \subseteq C(G)$ is an algebra with identity of functions on our compact G which is closed under conjugates and it seperates points:

$g \neq h \Rightarrow \exists f \in A$ st $f(g) \neq f(h)$, then A is dense in $C(G)$.

Here – take $A = \{$finite lin. combs of $\rho_{ij}^{\alpha}\}$

pf: A is closed under $\cdot$ (mult) because we can $\otimes$ reps.

A is closed under $\overline{}$ because we can $*$ reps. ↖ "bar" complex conjugation

A contains 1 since we have trivial rep.

$$\overline{\rho_{ij}}(g) = \rho_{ji}(g^{-1}) = \rho_{ij}^{*}(g)$$

ie) $\rho_{ij} \in A \Rightarrow \overline{\rho_{ij}} \in A$.

If G is a matrix group, (e.g. $SU(n)$,) then the fundamental representation (on $\mathbb{C}^n$) separates points. $\boxtimes$