5/15/01

From Quantum doubles to Frobenius Algs. & back.

If M is a monoid (assoc. mult & unit)
then the center
$Z_1(M) = \{a \in M \mid ab = ba \ \forall\ b \in M\} \subset M$ Z_1 commut.

If X is a set, we can consider $\{f : X \to X\} = Z_0(X)$ is a monoid.

① $Z_1(Z_0(X)) = \{id_X\}$ (the center of Z_0)

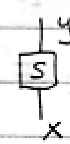
② $|Z_0(X)| = |X|^{|X|}$

sets = 0-categories
$\downarrow Z_0$
monoids
$\downarrow Z_1$
commutative monoid

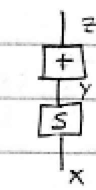
(1-)categories
$\downarrow Z_0$
$\otimes$-category (monoidal)
$\downarrow Z_1$
braided tensor categories
$\downarrow Z_2$
symmetric $\otimes$-category

If C is a category, $\mathrm{Hom}_C(x,y)$ = set of morphisms.

$s : x \to y$ $t \circ s : x \to z$
$t : y \to z$

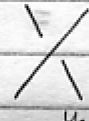


$t \circ s$:



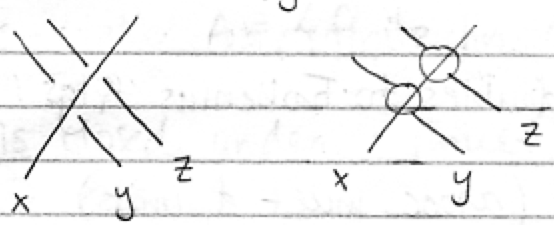
diagrams: bottom to top.

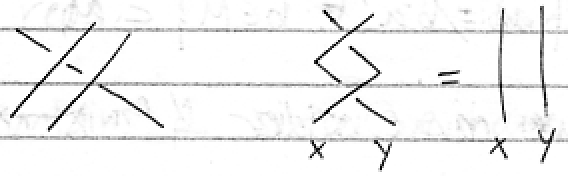
Note: $x \otimes y := xy$
$C_{x,y} : x \otimes y \longrightarrow y \otimes x$



$c_{x,y\otimes z}$

Braid relations

①  $= id_y \otimes c_{x,z} \otimes c_{x,y} \otimes id_z$

②  $\Leftrightarrow c_{y,x} \otimes c_{x,y} = id_{x\otimes y}$

iff C is symmetric.

close to being commutative. We can pull x away from y.

Claim:

a) $Z_2(Z_1(C))$ "$= 1$" for "all" $\otimes$-edges C,

b) $\dim Z_1(C) = (\dim C)^2$

(knowing the size of C, we know the size of the center)

Let C be a braided $\otimes$ category.

Define

$Z_2(C) =$ full $\otimes$ subcategory of C

w/

objects: $Obj(Z_2(C)) = \{X \in Obj\, C \mid c_{x,y} \circ c_{y,x} = id_{y\otimes x}\ \forall y \in Obj\, C\}$

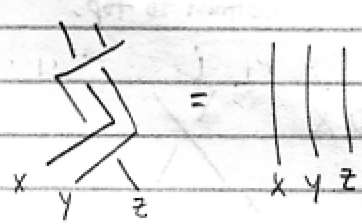
If $X, Y \in Z_2(C)$ (capital letters rep. objects)

"full" — specify what objects are

so $Hom_{Z_2(C)}(X,Y) = Hom_C(X,Y)$

① $c_{1,x} = c_{x,1} = id_x \Rightarrow 1 \in Z_2(C)$.

$X, Y \in Z_2(C) \Rightarrow X \otimes Y \in Z_2(C)$.



linear categories: $\mathbb{F}$ a field, $\forall$ pair of objects X, Y, $\mathrm{Hom}_C(X,Y)$ is a finite dim'l $\mathbb{F}$-v. space.

In lin. categories — makes sense to speak of direct sum
$= X, Y \in Z_2(C) \Rightarrow X \oplus Y \in Z_2(C)$.

ex) categories of rep. of a group. a linear group.
here hom — intertwiners

$X < Y$ (X is a subobject) (subrepresentation)
and $Y \in Z_2(C) \Rightarrow X \in Z_2(C)$.

Note: Center of a braided tensor category is a symmetric braided tensor category

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If G is a group, the category of representations, Rep G is a symmetric $\otimes$-category.

If H is a bi-algebra, then H-mod is $\otimes$-category.

$$(\pi_1 \times \pi_2)(a) := (\pi_1 \times \pi_2) \circ \Delta(a)$$

H a bialgebra is co-commutative iff $\Delta^{op} = \Delta$
where $\Delta^{op} = \sigma \circ \Delta$ where $\sigma(a \otimes b) = b \otimes a$.

So if G is a group, $\mathbb{F}$ a field, the group alg: $\mathbb{F}G$ is a co-commut. bialgebra

$$\Delta(g) = g \otimes g$$

If H is co-commut, $\Rightarrow$ H-mod is a symmetric $\otimes$-category.

$(*)\quad \pi_1 \otimes \pi_2 \simeq \pi_2 \otimes \pi_1$

- H is <u>quasi-co-commutative</u> if $\exists\, R \in H \otimes H$ invertible st $R\Delta(a)R^{-1} = \Delta^{op}(a)\ \forall a \in H$.

- H is <u>quasi-triangular</u> if $\exists\, R$ satisfying $(*)$ and

same as eqns on 4/24 the 2 conds. for quasitriangular

$$\left\{\begin{array}{l} (\Delta \otimes id)(R) = R_{13}R_{23} \\ (id \otimes \Delta)(R) = R_{13}R_{12} \end{array}\right\rangle \Rightarrow \boxed{R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}}$$

Young - Baxter Eqn.

$R_{12} = R \otimes 1 \in H^{\otimes 3}$

$R_{23} = 1 \otimes R$

So if H is finite dim'l Hopf algebra

$\Rightarrow D(H)$ "quantum double" of H

$X = (H^{op})^*$

$\leftarrow$ opposite Hopf alg. product is flipped.

Define:

$D(H) = X \otimes H$ as a <u>vector space</u> where

$f \otimes a$

$\dim_{\mathbb{F}} D(H) = (\dim H)^2$ (claim b) prev pg)

$\Delta(f \otimes a)$

If $\{e_i\}$ is a basis of H, we can define

$$R = \sum_i \underbrace{(1 \otimes e_i)}_{\in D(H)} \otimes \underbrace{(e^i \otimes 1)}_{\in D(H)}$$

$\{e^i\}$ dual basis in X.

and $R \in D(H) \otimes D(H)$

$\Rightarrow (D(H), R)$ is a quasitriangular Hopf alg.

$D(H)$-mod is braided and <u>not</u> symmetric if $\dim H > 1$.

Let H be a finite dim'l Hopf alg.
H-mod = finite left modules is a $\otimes$-category.
We can compute its center.

$$Z_1(H\text{-mod}) \simeq D(H)\text{-mod} \quad \forall H$$

equiv. as braided $\otimes$ categories

$Z_1(C)$ is defined for any category.
|| ← called the "center" of C.
"quantum double of C"

$Z_2(C)$ defined only for braided $\otimes$-categories.

C is a strict $\otimes$ category. We want to define $Z_1(C)$.

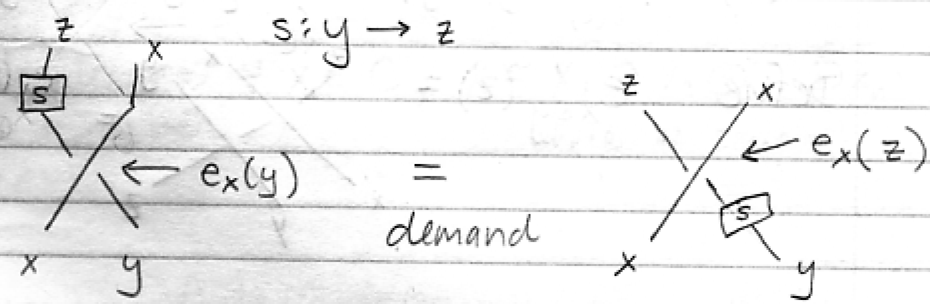
e_x: way to switch/commute w/ other elts. so you're in the center if you have a way to switch w/ other elts.

$\mathrm{Obj}(Z_1(C)) = \{ (X, e_X) \mid X \in \mathrm{Obj}\, C,\ e_X = \text{"half braiding for } X\text{"} \}$

$e_X(\cdot) : y \longmapsto e_X(y) \in \mathrm{Hom}_C(X \otimes y,\ y \otimes X)$ iso.

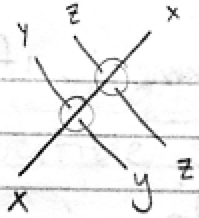
$y \in \mathrm{Obj}\, C$

satisfying: a) naturality



$\forall$ y, z and $s : y \to z$

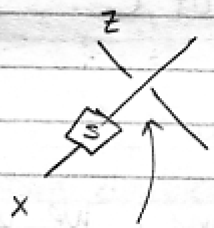
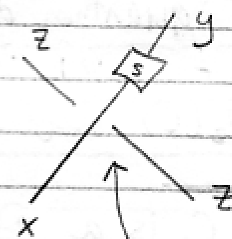
(we also require this in a braided monoidal category)

b) 

$e_x(y \otimes z) = id_y \otimes e_x(z) \circ e_x(y) \otimes id_z \quad \forall\, y, z \in \text{obj}\,C$

Now we must define what the morphisms are for any pair:

$\text{Hom}_{Z_1(C)}((x, e_x), (y, e_y))$

$= \{ s \in \text{Hom}_C(x,y) \mid$  $=$  $\}$

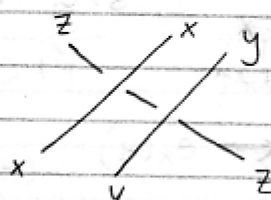
$e_y(z)$ $\qquad$ $e_x(z)$

$\forall\, z$

<u>Tensor unit</u>:

$\mathbb{1}_{D(C)} = (\mathbb{1}_C, e_1) \qquad e_1(x) = id_x \;\; \forall\, x.$

$(x, e_x) \underset{D(C)}{\otimes} (y, e_y) = (x \underset{C}{\otimes} y, e_{x \otimes y})$

Define $\quad e_{x \otimes y}(z) =$  $= e_x(z) \otimes id_y \circ id_x \otimes e_y(z) \quad \forall\, z.$

$\Rightarrow (x, e_x) \otimes \mathbb{1}_{D(C)} = (x, e_x) = \mathbb{1}_{D(C)} \otimes (x, e_x)$

Braiding what we call B. $\longrightarrow$ $C_{Z_1(C)}((x, e_x), (y, e_y)) = e_x(y) : x \otimes y \longrightarrow y \otimes x$

our: $B_{(x,e_x),(y,e_y)}$

since: $(x \otimes y) \otimes z = x \otimes (y \otimes z)$

monoidal category (really - only isomorphic)

Thm: $Z_1(C)$ is braided $\otimes$-category.

- If C is an $\mathbb{F}$-linear $\otimes$-category $\Rightarrow Z_1(C)$ is $\mathbb{F}$-linear.

C-category

- C has direct sums $\Rightarrow \forall\, x, y\ \exists\ x \oplus y$

  $Z_1(C)$ has direct sums

a pre-additive category

- If C is an abelian category $\Rightarrow Z_1(C)$ is abelian.

  (nothing to do w/ symmetric)

- If C has subobjects $\Rightarrow Z_1(C)$ has subobjects

  $\Updownarrow$

  $p \in End(x) \equiv Hom_C(x, x)$

  $p^2 = p \Rightarrow \exists\ y \in Obj\, C \quad s: x \to y,\ t: y \to x$

  $\cancel{st}\ t \cdot s = p$

  $s \cdot t = id_y$

- C is semisimple category $\Rightarrow Z_1(C)$ is semisimple.

We say an object $X \in C$ is simple iff

$End_C(X) \cong \mathbb{F}$

C semisimple $\Leftrightarrow \forall\, X \in C,\ X \simeq X_1 \oplus X_2 \oplus \dots \oplus X_n$

where X_i simple

Fact: H a finite dim'l Hopf alg.
TFAE:

① $D(H)$ is semisimple
② $D(H)^*$ is semisimple
③ H and H^* semisimple
④ $S_H^2 = id_H$ and $\dim H \neq 0$ in $\mathbb{F}$
($\operatorname{char} \mathbb{F} \nmid \dim H$)

In char 0, H s.s. iff H^* s.s. iff $S^2 = id$

Must prove $(X, e_X) \simeq (X_1, e_{X_1}) \oplus \dots \oplus (X_n, e_{X_n})$

$$\underset{\substack{\| \\ M_{i_1}(\mathbb{F}) \oplus \dots \oplus M_{i_n}(\mathbb{F})}}{\operatorname{End}_{Z_1(C)}((X, e_X))} \text{ is multi-matrix alg.}$$

$\operatorname{End}_{Z_1(C)}((X, e_X)) \subset \operatorname{End}_C(X)$

$E_{(X,e_X)} : \operatorname{End}_C(X) \longrightarrow \operatorname{End}_{Z_1(C)}(X, e_X)$

conditional expectations in prob. theory.

$E_{(X,e_X)}(a) = a$ if $a \in \operatorname{End}_{Z_1(C)}(X, e_X)$

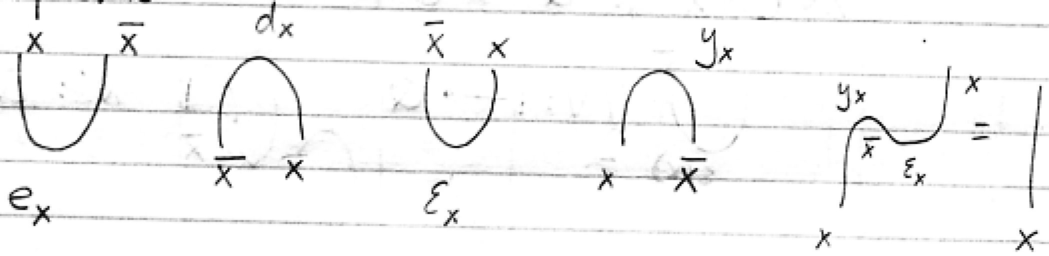
$E_{(X,e_X)}(abc) = a E_{(X,e_X)}(b) c$ if $a, c \in \operatorname{End}(X, e_X)$
$b \in \operatorname{End}(X)$.

$B \underset{E}{\longrightarrow} A$ Frobenius extension of A.

Prop: $A \subset B$ unital $\mathbb{F}$ algebras, $tr : B \to \mathbb{F}$, a non-degenerate trace st $tr\, ab = tr\, ba$ and $E : B \to A$ conditional expectation st $tr(E(a)) = tr(a)$.
If B is semisimple, then A is semisimple.

C - a $\otimes$-category, X an object.

Defn: $\overline{X} \in Ob(C)$ is a 2-sided dual of X iff $\exists$ morphisms



$$x \bigcirc \bar{x} = \bar{x} \bigcirc x = d(x) = d(\bar{x}) \in \mathbb{F}$$

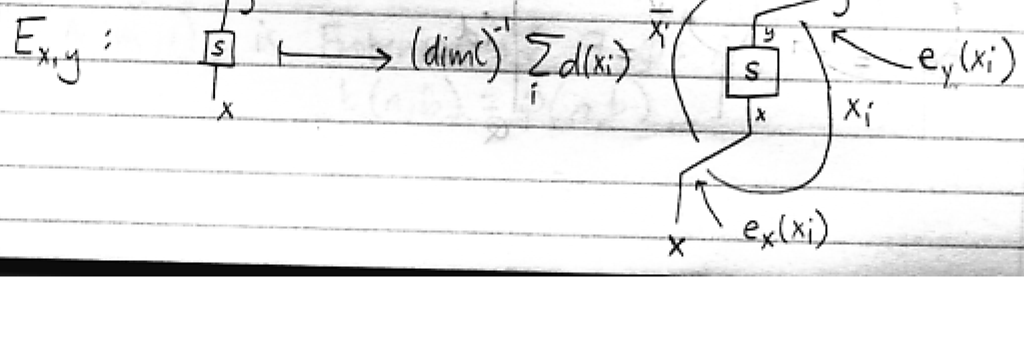
abstract notion of dimension

Let $(x, e_x), (y, e_y) \in Z_1(C)$

$$E_{x,y} : Hom_C(x,y) \longrightarrow Hom_{Z_1(C)}((x,e_x),(y,e_y))$$

$$\dim C = \sum_i d(x_i)^2 \neq 0$$

If H a s.s. Hopf alg, then $\dim H\text{-mod} = \dim_{\mathbb{F}} H$



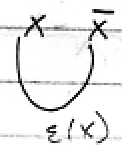
Claim: $\dim Z_1(C) = (\dim C)^2$

If H is a semisimple Hopf alg, $C = H\text{-mod}$
How do we prove this?

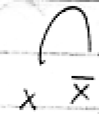
$Z_1(C) \underset{sph}{\simeq} B$, $\dim B = (\dim C)^2$

Let C be a spherical $\otimes$-category.

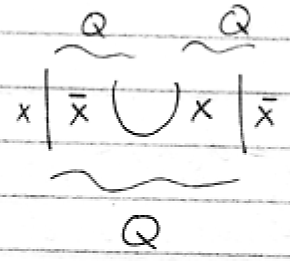
$X \in C$, $\bar{X}$, $Q = X \otimes \bar{X}$



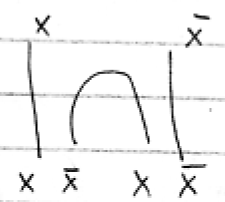
$v : 1 \longrightarrow Q$



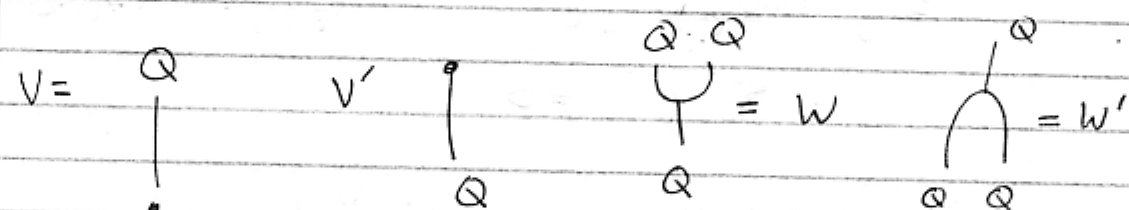
$v' : Q \longrightarrow 1$

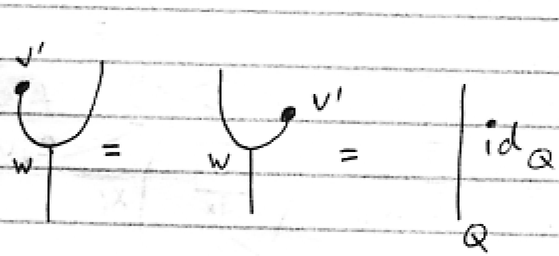


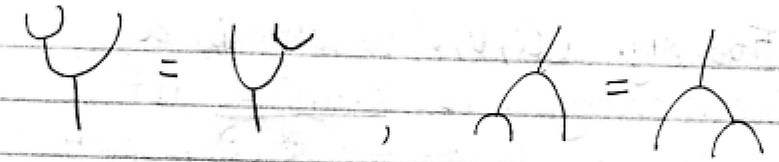
$W : Q \longrightarrow Q^2$

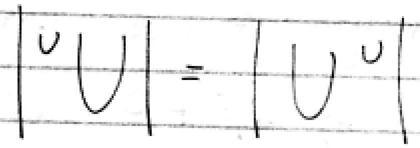


$W' : Q^2 \longrightarrow Q$



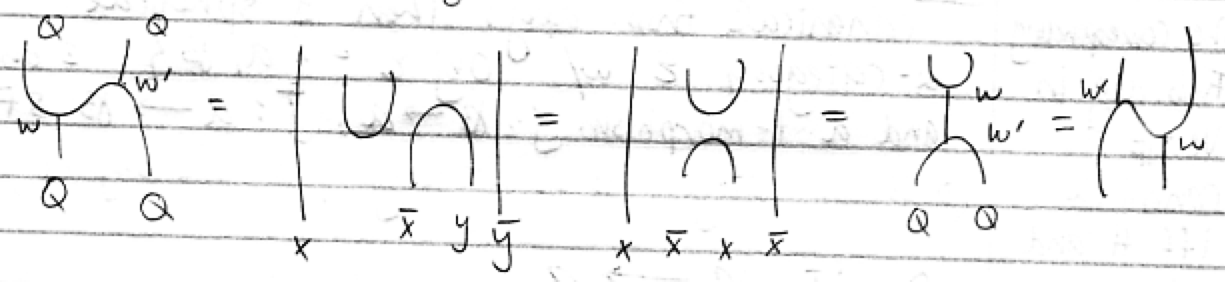




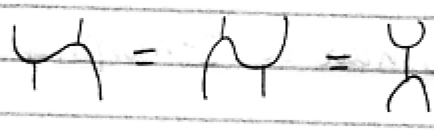


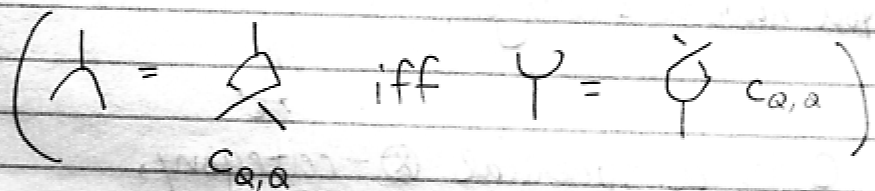
(A) (Q, v, w') is an <u>algebra</u> in C

(B) (Q, v', w) is a <u>coalgebra</u> in C



<u>Defn</u>: (Q, v, v', w, w') is a <u>Frobenius</u> <u>algebra</u> in C if (A), (B) and





<u>Thm</u>: There is a 1-1 correspondence between Frob. algebras (in the above sense) in $Vect_{\mathbb{F}}$ and Frob. algebras (in the usual sense).

$(A, m, 1)_{\mathbb{F}}$ is Frobenius if $\exists\ \phi : A \longrightarrow \mathbb{F}$ st
$$b(a,b) = \phi(a,b)$$

v', v $= \bigcirc = d(x)$ $\quad$ $= |\bigcirc| = ||\, d(x)$

<u>Defn</u>: A Frob. alg. (Q, v, v', w, w') in a $\mathbb{F}$-linear category is <u>canonical</u> iff

$$= v' \circ v = \alpha \, id_{1}$$

$$= w' \circ w = \beta \, id_{Q} \quad w/ \ \alpha\beta \neq 0$$

Q

<u>Fact</u>: Every Hopf alg. is Frob.

↓

The corresponding Frob. alg in $Vect_{\mathbb{F}}$ is canonical iff H and H^* are semisimple.

<u>Thm</u>: Let $\mathcal{A}$ be a spherical (Q, v, v', w, w') canonical Frob. alg. Then $\exists$ spherical 2-category $\mathcal{E}$ w/ $Obj\, \mathcal{E} = \{a, \beta\}$ 2 objects. and a 1-morphism $j : \beta \to a$, $\bar{j} : a \to \beta$ st

(A tensor category is a 2-category w/ one object)

$Q = j\bar{j} : a \longrightarrow a$

same for id_a.

So we have a 2-morphism from id_a to $j\bar{j}$

If you have a 2 category, $Hom(a,a)$ is a $\otimes$ category.
$\cong \mathcal{A}$

If C is a spherical $\otimes$-category, define

$\mathcal{A} = C \boxtimes C^{op}$ ← morphisms go other way

$\dim \mathcal{A} = (\dim C)^2$

(Q, v, v', w, w') canonical Frob. alg in $\mathcal{A}$

$\rightsquigarrow$ apply machinery, get a 2-category $\mathcal{E}$

$B = Hom_{\mathcal{E}}(\mathcal{B}, \mathcal{B})$

$\int$ || equiv. as a $\otimes$ category

$Z_1(C)$.

$\dim B = \dim A$

* This shows $Z_1(C)$ is a nontrivial category.