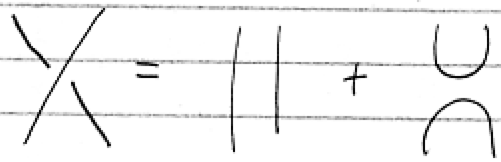
5/22/01

Redo 1st quarter's work on spin networks replacing

Binor Identity: spin 1/2 skein rel.

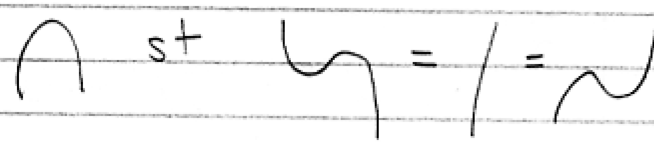


where lines are labelled by $V = \mathbb{C}^2$ and

$\cup$ is $\omega : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}$ the symplectic structure

$\omega(e_1 \otimes e_2) = 1 \qquad \omega(e_1 \otimes e_1) = 0$
$\omega(e_2 \otimes e_1) = -1 \qquad \omega(e_2 \otimes e_2) = 0$

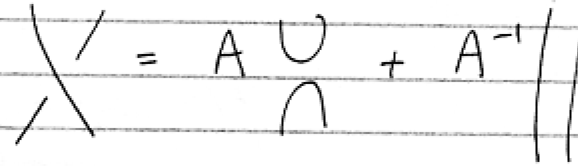
which determines $\cap$ s.t.

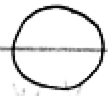


replace w/

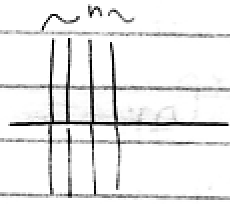
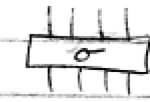
$0 \neq A \in \mathbb{C}$

Also –



 $= d = -(A^2 + A^{-2})$

In the 1st quarter we defined "symmetrizers"

$P_s =$  $= \frac{1}{n!} \sum_{\sigma \in S_n}$ 

perm. group

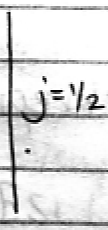
This $P_s : V^{\otimes n} \longrightarrow V^{\otimes n}$ has

$P_s^2 = P_s$ so we let $j = S^n V = \text{Range } P_s$

where j is a number, related to n. $\boxed{2j = n}$

$j = 0, \frac{1}{2}, 1, \ldots$
$n = 0, 1, 2, \ldots$

dim space $= 2j+1$

Ex)  1 strand n. so $j = \frac{1}{2}$

Everything is built from $V = \mathbb{C}^2$ and symplectic structure ω on it.

The linear transf that preserve ω are precisely those w/ $\det = 1$.

We saw

$$\omega(gv \otimes gw) = \omega(v,w) \qquad \forall\ v,w \in V$$

iff $g: \mathbb{C}^2 \to \mathbb{C}^2$ (linear) has determinant 1.

ie) $g \in SL(2,\mathbb{C})$.

So – $SL(2,\mathbb{C})$ acts as symmetries.

(rep. on groups, we can then tensor them)

So – $SL(2,\mathbb{C})$ has a representation on $V = \mathbb{C}^2$ and $V^{\otimes n}$ gets a rep of $SL(2,\mathbb{C})$ since we can tensor reps.

And –
$\sigma: V^{\otimes n} \to V^{\otimes n}$ are intertwining operators, or "intertwiners", for all $\sigma \in V^{\otimes n}$ so that

$$p_s = \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \text{ is an intertwiner!}$$

So, $j = \text{Range } p_s \subseteq V^{\otimes n}$ is a subrepresentation.

$$\psi \in \text{Range } p_s \text{ iff } \psi = p_s \phi \quad \exists\ \phi \in V^{\otimes n}$$

We used to say $\rho(g)\psi$ now, drop the name ρ.

Here we're being sloppy. We're writing gv instead of $\rho(g)v$. → iff $g\psi = g p_s \phi \quad g \in SL(2,\mathbb{C})$

iff $g\psi = p_s g \phi$ since p_s is an intertwiner.

Recall: $f: V \longrightarrow W$ f is an intertwiner if
$\rho \qquad \rho'$ $f\rho(g) = \rho(g) f$.

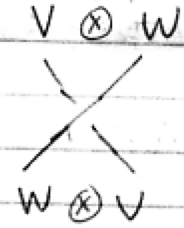
Again:

$$\psi = p_s \phi$$
iff
$$g\psi = g p_s \phi$$
iff
$$g\psi = p_s g \phi$$
iff
$$g\psi \in \text{Range } p_s$$

p_s is an intertwiner

---

when V, W reps of groups



this map $\begin{array}{c} V \otimes W \\ \downarrow \\ W \otimes V \end{array}$ is an intertwiner

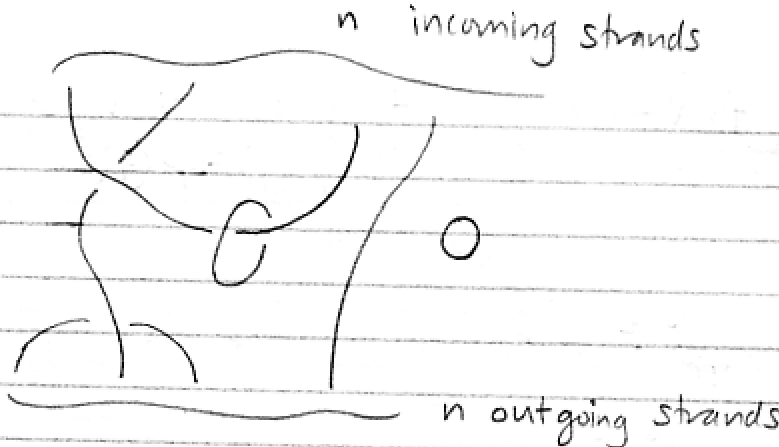
---

We call $j \subseteq V^{\otimes n}$ the "spin-j representation" of $SL(2, \mathbb{C})$. And in particular, $\frac{1}{2} = V$ is called the "spin-$\frac{1}{2}$ representation" of $SL(2, \mathbb{C})$.

__Thm:__ Every irreducible (finite dim'l) representation of $SL(2, \mathbb{C})$ is equivalent to one of these spin-j reps where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ and $\dim(j) = 2j + 1$.

Now, let's make up "q-deformed" symmetrizers analogous to p_s, but for Kauffman bracket relations.

tangle:



gives us an operator from $V^{\otimes n} \longrightarrow V^{\otimes n}$

$V = \mathbb{C}^2$, but

$\cap$, $\cup$, $\times$, $\times$ are operators

st

$$\times = A \, || + A^{-1} \, \underset{\cap}{\cup}$$

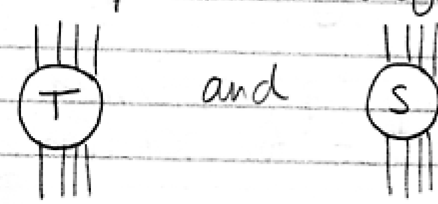
$$O = -(A^2 + A^{-2})$$

Any tangle w/ n strands in, n strands out gives an operator

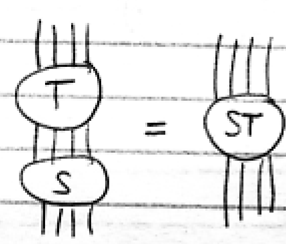
$T: V^{\otimes n} \longrightarrow V^{\otimes n}$.

Linear combinations of these operators form, not only a vector space, but also an algebra where multiplication is defined:

tangles



The product is:



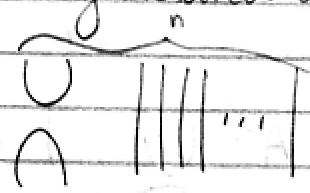
$S \circ T$

do T 1st

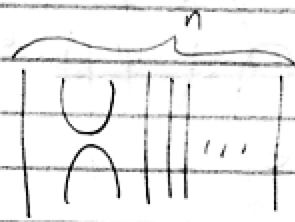
This algebra formed by linear combinations of our linear operators is called the <u>Temperley-Lieb alg</u>, TL_n.

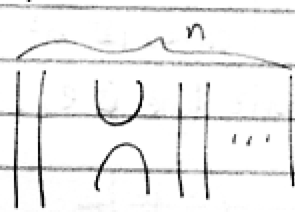
Now, we want to find "q-deformed symmetrizers" $p \in TL_n$.

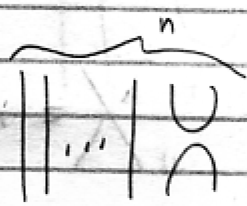
TL_n is generated as an algebra by $e_1, \dots, e_{n-1}$

$e_1 =$ 

cap, cup takes up 2 strands.

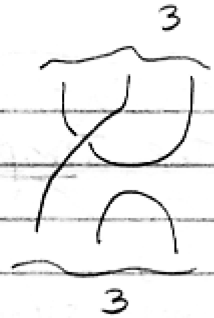
$e_2 =$ 

$e_3 =$ 

$e_{n-1} =$ 

Any elt of <u>Temperley-Lieb alg</u> ("TL_n") is a linear comb. of products of $e_1, \dots, e_{n-1}$.

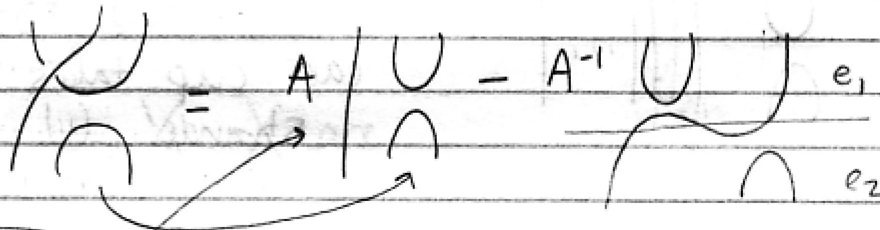
ie) TL_n consists of lin. combs of products of these e_i.

ex) 

So in TL_3. Want to express this as a lin comb of e_i.

We can use skein relation.

.by Kauffman bracket skein relation

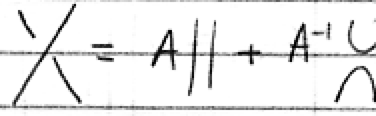


 $= Ae_2 + A^{-1}e_2e_1$

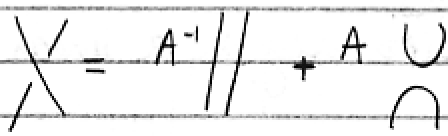
↑ comes 1st at top

(above horiz line is e_1), below horiz line is e_2.

<u>Recall:</u>

Kauffman bracket skein rel: 

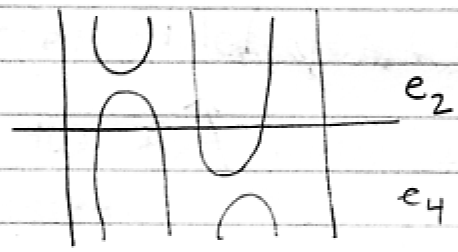
Also can use:

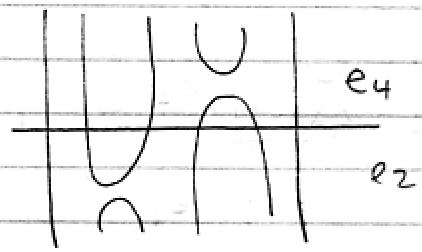


 $= -(A^2 + A^{-2})$

These e_i satisfy some relations:

$$e_i e_j = e_j e_i \text{ if } |i-j| \geq 2$$

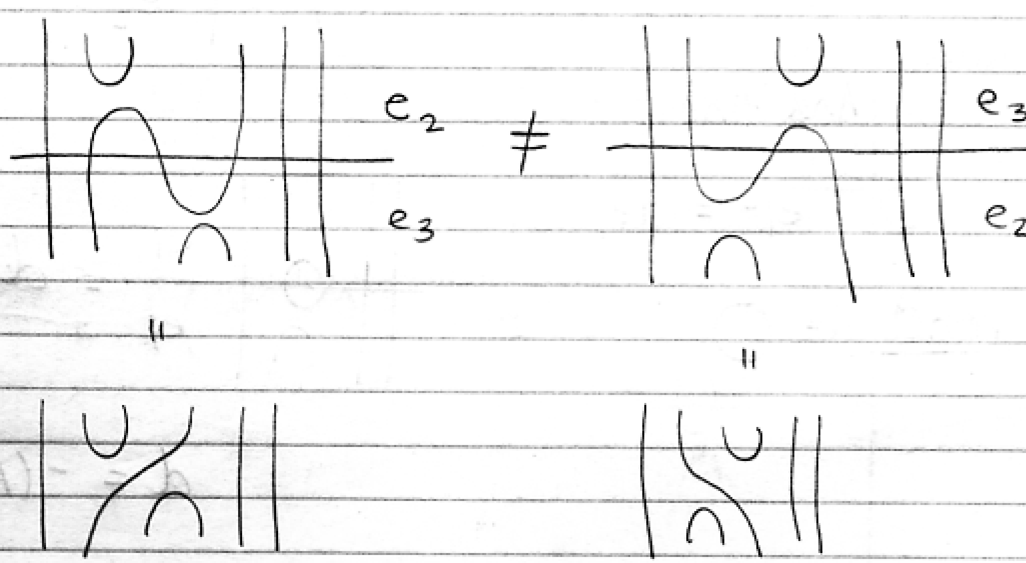




they commute since not tangeled up in one another

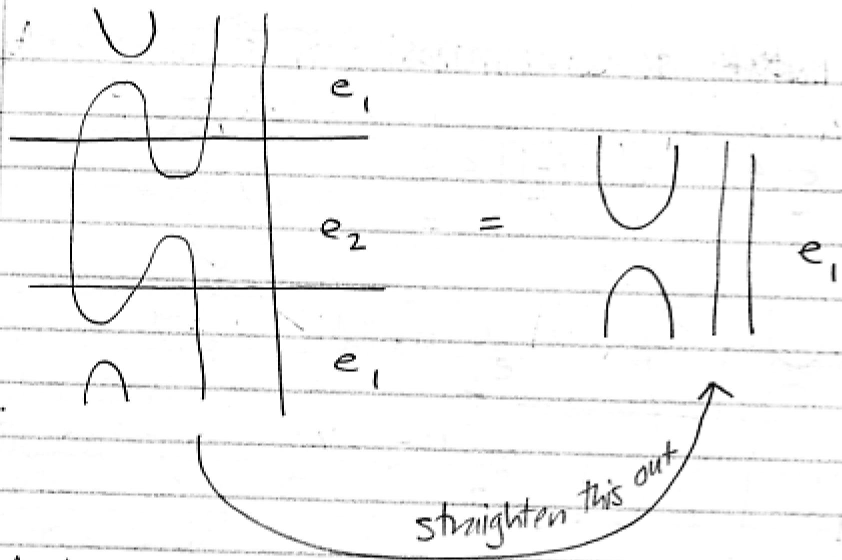
we can stretch things to make these pictures look like each other.

But:



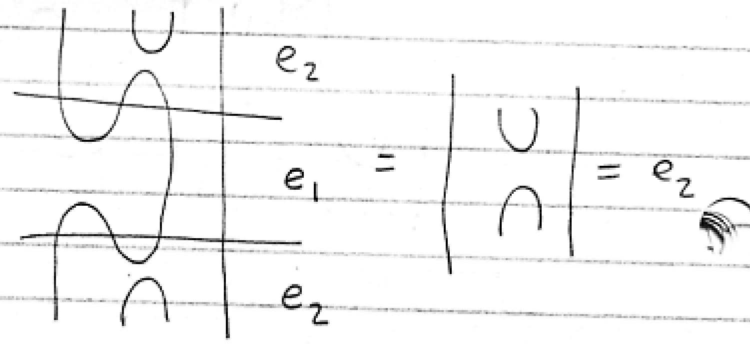
So, we don't have $e_i e_j = e_j e_i$ if $\boxed{|i-j| = 1}$.

<u>Instead</u>: we have (1) $\boxed{e_i e_{i+1} e_i = e_i}$



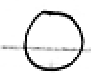
And—

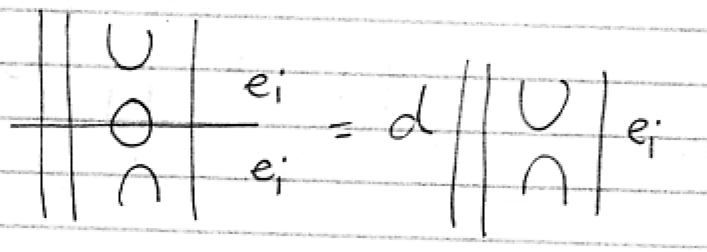
② $\boxed{e_{i+1}\, e_i\, e_{i+1} = e_{i+1}}$



And finally—

③ $e_i^2 = d e_i$

<u>recall</u> $d=$ 



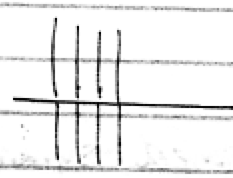
$$d = -(A^2 + A^{-2})$$

Thm: TL_n is the algebra freely generated by e_i modulo precisely these relations:

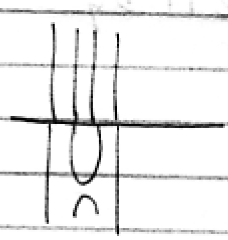
$$e_i e_j = e_j e_i \qquad |i-j| \geq 2$$

$$e_i e_{i+1} e_i = e_i, \quad e_{i+1} e_i e_{i+1} = e_i, \quad e_i^2 = d e_i$$

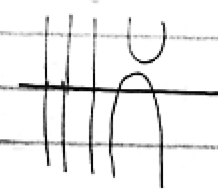
In the first quarter, we saw $p_s : V^{\otimes n} \longrightarrow V^{\otimes n}$ satisfied:



symmetrizer & stick in a cap, cup,



$= 0$ ✓ and

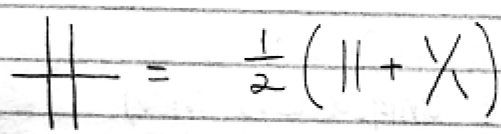


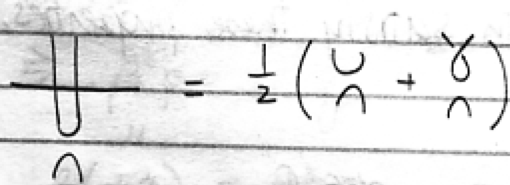
$= 0$ ✓

(having cup above)

Recall:

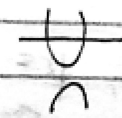
Ex)





but

so that



$= 0.$

Let's generalize this to the new context:

Thm: (insert: If A is not a root of unity: ie) $A^k \neq 1 \quad \forall\ k = 1, 2, \ldots$) For all n, $\exists$ a unique nonzero $p \in TL_n$ st

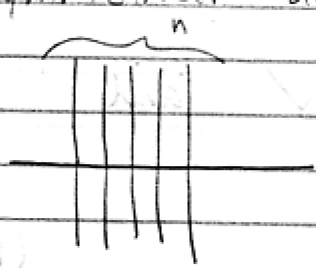
so p is linear

① $p^2 = p$ (projection operator)

② $pe_i = 0 \quad \forall\ i \leq n-1$

③ $e_i p = 0 \quad \forall\ i \leq n-1$

We call p the "q-deformed symmetrizer" or "q-symmetrizer" and denote it by



<u>proof</u>: First we'll prove uniqueness then existence.

Suppose p and q both satisfy these properties. Want to show $p = q$.

Write $p = \alpha \mathbb{1} + u$, $\alpha \in \mathbb{C}$

product of no e_i's

where u is a finite linear comb. of nontrivial products of e_i's, since TL_n is generated by e_i's.

Similarly, write

$$q = \beta 1 + v$$

Then

$$pq = p(\beta 1 + v)$$

$$= \beta p$$

* But $p(e_i) = 0$ by one of properties of p.

$pv = 0$ since v is a lin. comb of products of e_i and $pe_i = 0$ $\forall i$.

But

$$pq = (\alpha 1 + u) q$$

$$= \alpha q$$

since $e_i q = 0$ $\forall i$ so $uq = 0$ since u is a comb of products of e_i.

Thus, $\beta p = \alpha q$.

Now, use ① the fact that p and q are projections. So,

$$\beta^2 p^2 = \alpha^2 q^2$$

$p^2 = p$, $q^2 = q$ by property ①

$$\Rightarrow \beta^2 p = \alpha^2 q$$

$$\beta(\alpha q) = \alpha^2 q$$

and $q \neq 0$ $\Rightarrow$ $\beta\alpha = \alpha^2$

$\Rightarrow$ $\alpha = \beta$

But $\alpha = \beta = 1$ since $p^2 = p$.

ie)

$$pp = p(\alpha 1 + u) = p$$

but $pu = 0$ since $pe_i = 0$

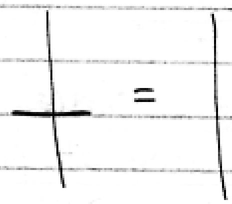
$$\Rightarrow \alpha p = p$$

$\Rightarrow \alpha = 1$ (since $p \neq 0$) Similarly $\beta = 1$.

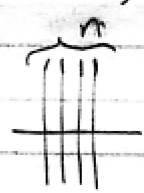
Thus, $pq = p$ and $pq = q$

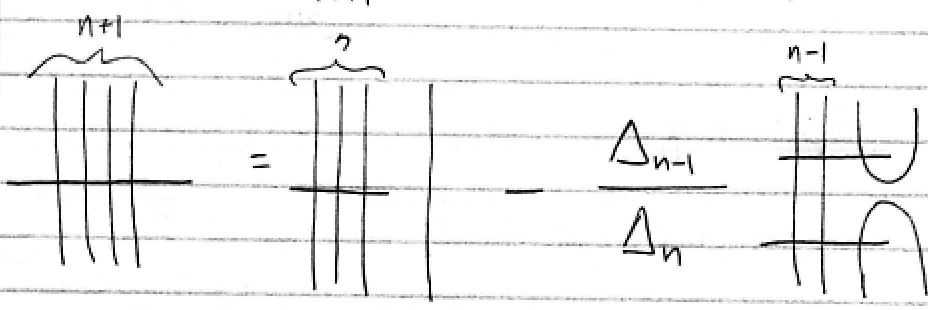
so $p = q$. ✓

<u>Existence:</u> For $n=1$

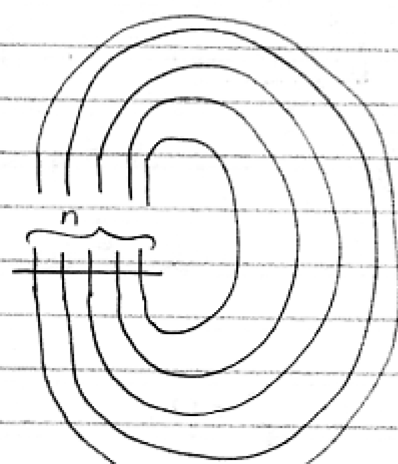


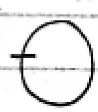
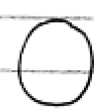
does the job. ie) $1 \in TL_1$ is our p for $n=1$.

We define  for higher n recursively:



where

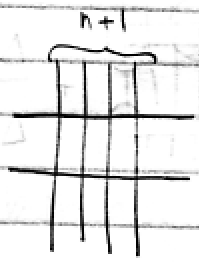
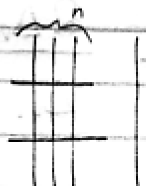
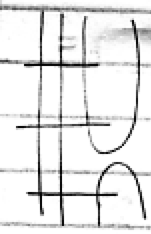
$\Delta_n =$ 

ex) $\Delta_1 =$  $=$  $= d = -(A^2 + A^{-2})$

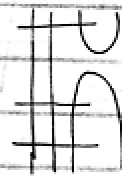
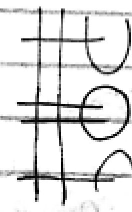
Let's show by induction that its a projection operator.
(ie. $p^2 = p$)

$$(a-b)^2 = a^2 - ba - ab + b^2 \quad a,b \text{ not commut.}$$

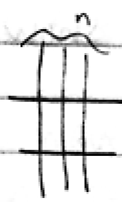
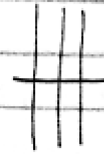
$n+1 = 4$
$n = 3$
$n-1 = 2$

 $=$  $- \dfrac{\Delta_{n-1}}{\Delta_n}$ 

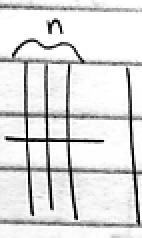
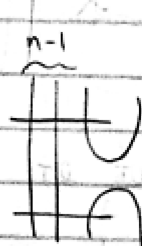
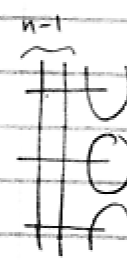
p^2

$- \dfrac{\Delta_{n-1}}{\Delta_n}$  $+ \left(\dfrac{\Delta_{n-1}}{\Delta_n}\right)^2$ 

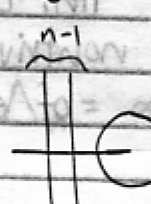
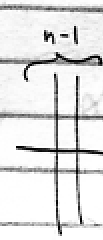
Now we can use the inductive hypothesis:

 $=$ 

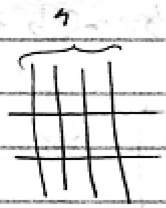
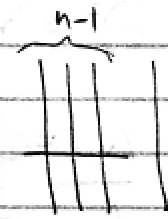
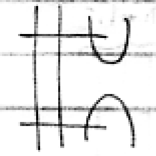
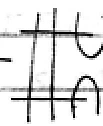
top eqn.

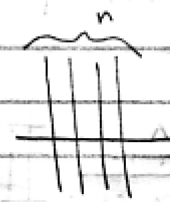
$=$  $- \dfrac{2\Delta_{n-1}}{\Delta_n}$  $+ \left(\dfrac{\Delta_{n-1}}{\Delta_n}\right)^2$ 

We need to get rid of the 2 and circle.

<u>Lemma:</u>  $= \dfrac{\Delta_n}{\Delta_{n-1}}$ 

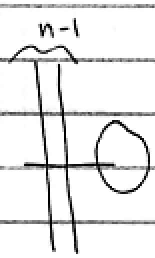
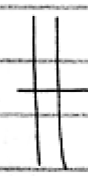
Using the lemma, we get:

$p^2 =$  $=$  $- \dfrac{2\Delta_{n-1}}{\Delta_n}$  $+ \dfrac{\Delta_{n-1}}{\Delta_n}$ 

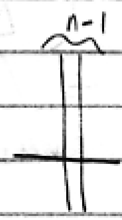
$=$ by defn  ☒

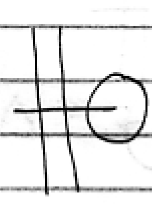
So we need only to prove the lemma.

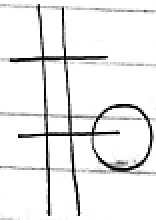
<u>proof of lemma:</u> Hard part is to show

 $= \alpha$ 

As in the proof of uniqueness,

$p =$  $= 1 + u$ where u is a lin. comb. of nontrivial products of e_i's.

$q =$  $= \alpha 1 + v$ where $\alpha \in \mathbb{C}$ and v is a lin. comb of nontrivial products of e_i's.

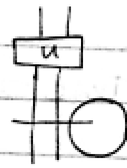
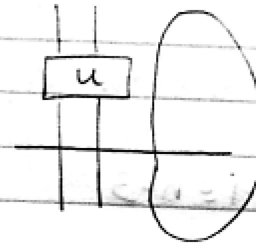
 $= qp = (\alpha 1 + v)p$

$= \alpha p$ since $vp = 0$, since $e_i p = 0 \ \forall i$

but also:

$$qp = q(1+u) = q$$

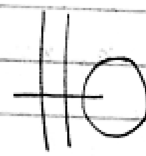
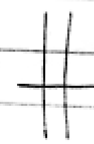
since

$qu =$  $=$  $= 0$ by the inductive hyp.

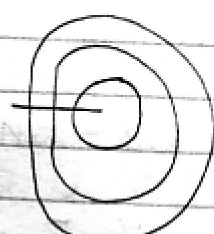
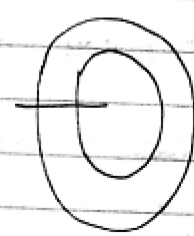
- u has e_i's in it

so we have $\alpha p = q$, which is what we were trying to show.

Need to find α:

 $= \alpha$ 

$\Rightarrow$

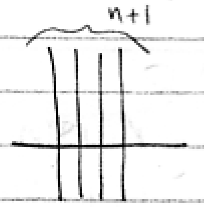
 $= \alpha$ 

$$\Delta_n = \alpha \Delta_{n-1} \quad \text{so, } \alpha = \Delta_n / \Delta_{n-1}.$$

$\boxtimes$

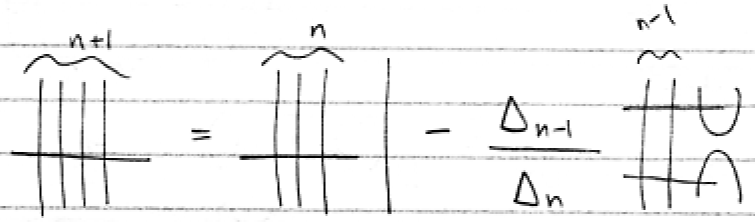
end lemma

Now – why is



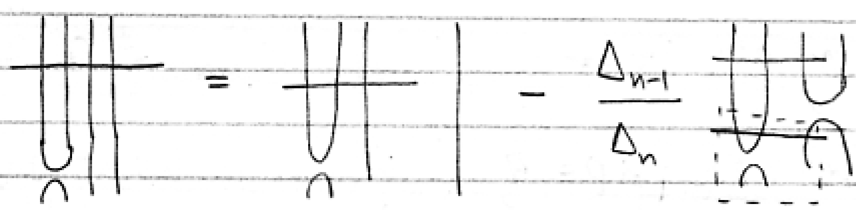
times e_i (in either order) equal to zero?

Recall:



Inductive hyp: Assume true for n, prove true for $n-1$.

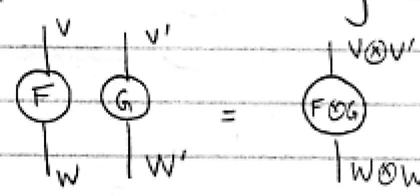
If $1 \leq i \leq n-3$



" 0 by hyp.   " 0 by hyp.   what's in box covered by inductive hyp

$e_i p = 0$
(and similarly $p e_i = 0$)

Note: recall sticking side by side – tensoring



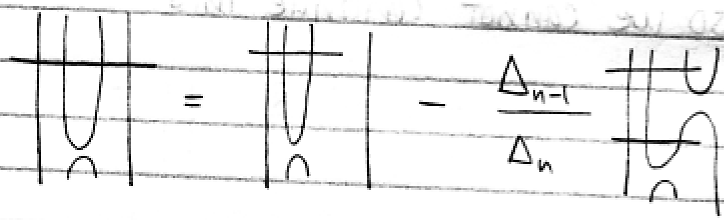
and if F or G are zero, then ⊗ is too.

If F or G $=0$, then $F \circ G = 0$ & $F \otimes G = 0$.

* all our pictures represent multiplication (composition or tensoring) so if a part is zero, all is zero.

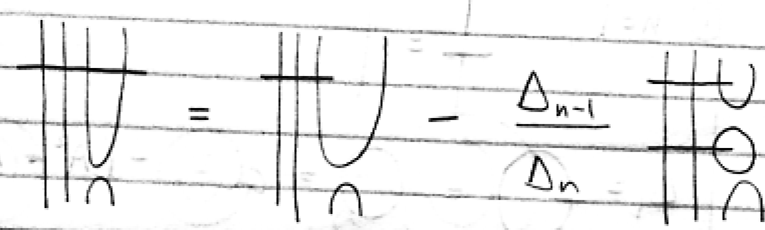
Now, if $i = n-2$

$e_i p = 0$

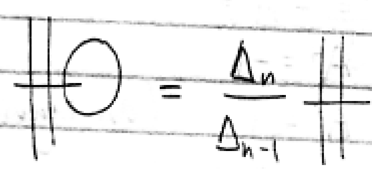


same result.

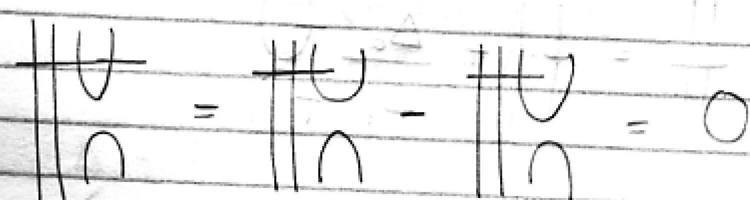
If $i = n-1$



But our lemma says:



so we get (from above)



so $e_i p = 0$ and similarly $p e_i = 0$.

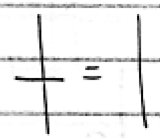
Note: we haven't used hypothesis that A isn't a root of unity.

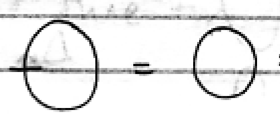
* Claim: If A isn't a root of unity, then Δ_n is never zero so this argument is fine. (we divide by Δ_n everywhere).

continued claim:

If A is a root of unity, for some n, Δ_n will be zero so we cannot continue this recursion. ∎

Examples:

① $n=1$, 

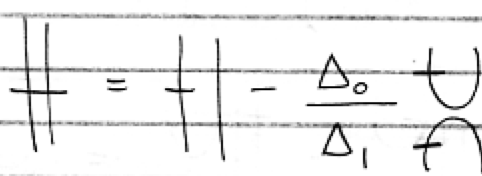
$\Delta_1 =$  $= d = -(A^2 + A^{-2})$

So, $\Delta_1 = 0$ if $A^2 = -A^{-2}$

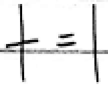
ie. $A^4 = -1$

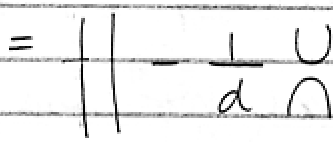
ie. $A^8 = 1$ (an eighth root of unity)

② $n=2$



recall $\Delta_0 = 1$ (empty picture is one)



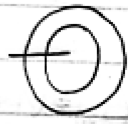
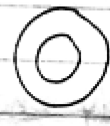
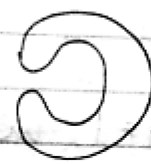


Let's check that this is a projection operator.

$$\# = \| - \frac{2}{d}\overset{\cup}{\cap} + \frac{1}{d^2}\overset{\cup}{\underset{\cap}{\circ}}$$

$$= \| - \frac{2}{d}\overset{\cup}{\cap} + \frac{1}{d}\overset{\cup}{\cap}$$

$$= \| - \frac{1}{d}\overset{\cup}{\cap}$$

So, $\Delta_2 =$  $=$  $- \frac{1}{d}$ 

$$= d^2 - \frac{1}{d}(d)$$

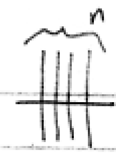
$$= d^2 - 1$$

$$= (A^2 + A^{-2})^2 - 1$$
$$= A^4 + 1 + A^{-4}$$

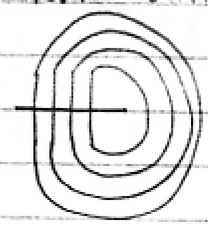
So: $\Delta_0 = 1$

$\Delta_1 = -(A^2 + A^{-2})$

$\Delta_2 = A^4 + 1 + A^{-4}$

$\boxed{\text{HW}}$: Use the recursive defn. of  to get

a recursion relation for

$\Delta_n =$  (close up projection operators)

Use this to get a closed form formula for Δ_n.

Guess: $\Delta_3 = -(A^6 + A^2 + A^{-2} + A^{-6})$??

When $A=1$, we already know $\Delta_n = (n+1)(-1)^n$

bozon/fermion distinction

when $A \neq 1$, we say

$\Delta_n = [n+1]_q (-1)^n$ where $[n+1]_q$ is called a "q-integer"