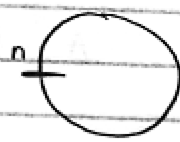
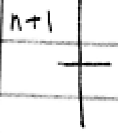
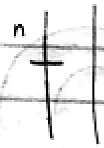
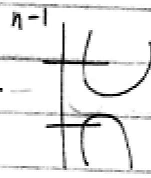
5/29/01

Homework Review

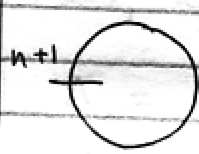
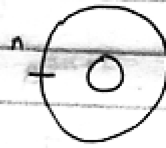
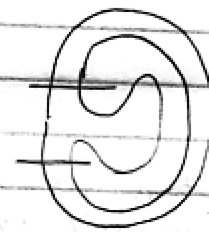
$\Delta_n =$ 

$n = \#$ of strands
$j = n/2$

 $=$  $-$ $\dfrac{\Delta_{n-1}}{\Delta_n}$ 

relation on 

$\Rightarrow$ closing them up:

 $=$  $-$ $\dfrac{\Delta_{n-1}}{\Delta_n}$ 

" 

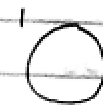
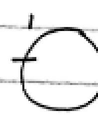
$$\Rightarrow \Delta_{n+1} = \Delta_n \cdot (A^2 + A^{-2}) - \frac{\Delta_{n-1}}{\Delta_n} \Delta_n$$

$$\Rightarrow \Delta_{n+1} = -\Delta_n (A^2 + A^{-2}) - \Delta_{n-1}$$

$\Delta_0 = 1$

$\Delta_1 = -(A^2 + A^{-2})$

<u>Recall:</u>

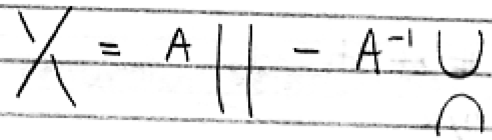
 $=$  so  $=$ 

$= \Delta_1 = -(A^2 + A^{-2})$

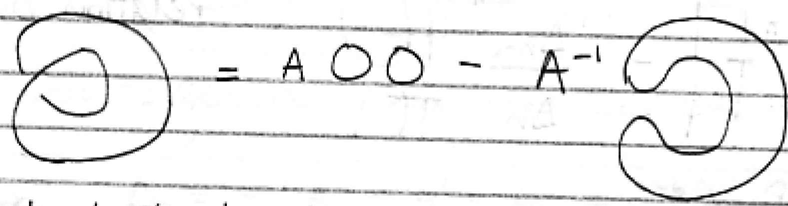
<u>Recall:</u> tensoring $\&$, composing are linear in each component

ie) $f \circ (g + h) = f \circ g + f \circ h$

Skein Relation:



close up:



Must check this is an equality.

HW — cont.

Guess:

$$\Delta_n = (-1)^n \left(A^{2n} + A^{2n-4} + \dots + A^{-2n}\right)$$

(a geometric series!)

$$(*) \quad = (-1)^n A^{-2n}\left(A^{4n} + A^{4n-4} + \dots + A^4 + 1\right)$$

Recall: $1 + x + \dots + x^n = S$ $\quad$ $S = \text{sum}$

$x + \dots + x^{n+1} = xS$

subtract: $1 - x^{n+1} = (1-x)S$

so $S = \dfrac{x^{n+1} - 1}{x - 1}$

So our above sum $\left(A^{4n} + A^{4n-4} + \dots + A^4 + 1\right)$

$$= \frac{A^{4(n+1)} - 1}{A^4 - 1}$$

So, $(*)$ becomes

$$(-1)^n A^{-2n} \quad \frac{A^{4(n+1)}-1}{A^4-1}$$

$$= (-1)^n \frac{A^{2n+4} - A^{-2n}}{A^4-1}$$

divide by A^2

$$= (-1)^n \frac{A^{2n+2} - A^{-(2n+2)}}{A^2 - A^{-2}}$$

Now, let $q = A^2$. Then

$$\Delta_n = (-1)^n \frac{A^{2n+2} - A^{-(2n+2)}}{A^2 - A^{-2}}$$

$$= (-1)^n \frac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}}$$

We define the "q-integer"

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

<u>Note</u>: As $q \to 1$, this approaches n.

So $\Delta_n = (-1)^n [n+1]_q$.

$-(A^2+A^{-2})$

Check $\Delta_{n+1} = \Delta_n \Delta_1 - \Delta_{n-1}$

or:

$$(-1)^{n+1}\frac{q^{n+1}-q^{-(n+1)}}{q-q^{-1}} = (-1)^n \frac{q^n-q^{-n}}{q-q^{-1}} \cdot -(q+q^{-1})$$
$$- (-1)^{n-1}\frac{q^{n-1}-q^{-(n-1)}}{q-q^{-1}}$$

Divide by -1

$$(q^{n+1}-q^{-(n+1)}) = (q^n-q^{-n})(q+q^{-1})$$
$$- q^{n-1} - q^{-(n-1)}$$
$$= q^{n+1} - q^{-(n+1)} + q^{n-1} - q^{-(n-1)}$$
$$- q^{n-1} - q^{-(n-1)}$$

check! ✓

So — the quantum dimensions — are just the q-integers

Recall — defining $p_{n+1} \in TL_{n+1}$ (t. lieb alg) via our recursion relation involves dividing by Δ_n, so p_{n+1} will exist (and have properties listed in Theorem) if $\Delta_n \neq 0$,

$$\Delta_n = (-1)^n \frac{q^{n+1}-q^{-(n+1)}}{q-q^{-1}}$$

if $n+1 < r$ then p_{n+1} exists

So, $\Delta_n = 0$ iff $q^{n+1} = q^{-(n+1)}$

$$q^{2(n+1)} = 1$$

So — p_n exists for all n if q is not a root of unity.

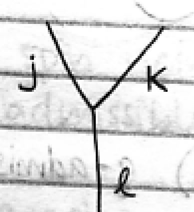
If q is a primitive $2r^{\text{th}}$ root of unity (i.e. $q^{2r}=1$ and $q^k \neq 1$ if $k<2r$) then we can not define p_r, but we can define $p_0, \ldots, p_{r-1}$ using the recursion relation.

Translating from $n = 0, 1, 2, \ldots$ to $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ $\quad j = \frac{n}{2}$

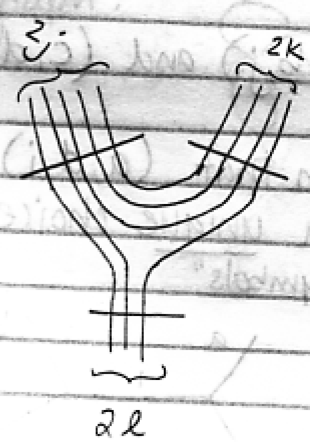
we have: the "q-admissible" spins j, i.e. those for which p_n is defined $(n = 2j)$ via recursion are

$$j = \underbrace{0, \tfrac{1}{2}, \ldots, \tfrac{r}{2} - \tfrac{1}{2}}_{r \text{ of these}}$$

Now — copying our work from 1^{st} quarter, we define



via



which we can do uniquely iff

① $j + k + \ell \in \mathbb{N}$ (parity)

② Δ-inequality $\quad j + k \geq \ell$, $\quad k + \ell \geq j$, $\quad \ell + j \geq k$

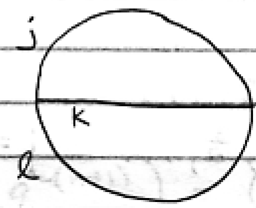
We call a triple (j, k, ℓ) satisfying ① & ② <u>admissible</u>.

We call (j, k, ℓ) <u>q-admissible</u> if they're admissible and

3) $j, k, \ell \leq \frac{r-1}{2}$

4) $j + k + \ell \leq r - 1$

<u>Thm:</u>

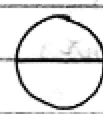


$\neq 0$

iff

(j, k, ℓ) is q-admissible.

Kauffman's argument in "Temperley-Lieb Recoupling Theory" proceeds by

1) getting recursion formula for 

2) solving it in closed form.

Let $AD_q = \{(j, k, \ell) \mid (j, k, \ell) \text{ q-admissible}\}$

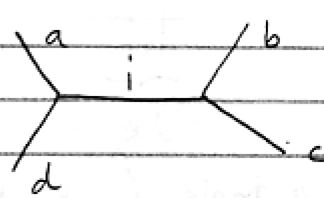
<u>Recoupling Thm:</u> If (a, b, j) and $(c, d, j) \in AD_q$

where we sum over i such that $(a, d, i), (b, c, i) \in AD_q$

then holds for a <u>unique</u> choice of "quantum 6j symbols"

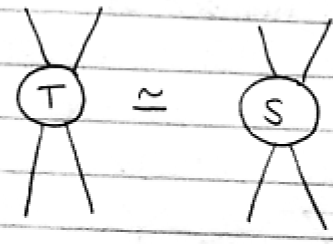
6-j symbols allow us to express one in terms of other

$\simeq \sum_i \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}_q$



q-deformed 6-j symbols

Where we say

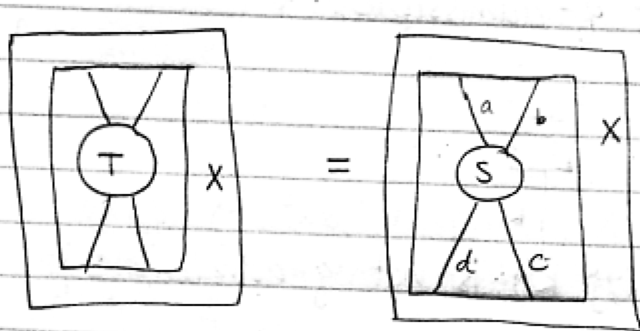


S, T are linear comb. of tangles modulo Kauffman bracket skein relations.

T & S are equivalent when you close them up in any way, we get the same #.

if

closed spin network:



∀ X want this equality to hold.

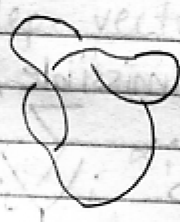
X ranges over spin networks w/ loose ends labelled by a, b, c, d

(T & S are equiv. if these closed spin networks are the same)

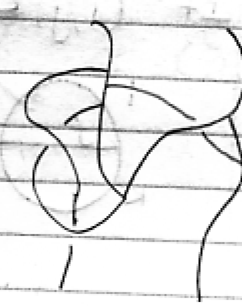
———— (end of statement of thm) ————

Recall – spin network, edges labelled by spins that are admissible (trivalent graph)

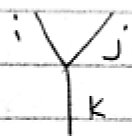
closed graph:



open graph: loose ends

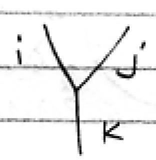


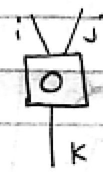
Example: We can similarly define $\simeq$ for spin networks w/ any number of loose ends, and then



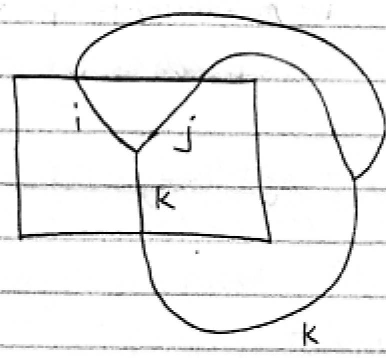
$\simeq 0$ ← zero operator.

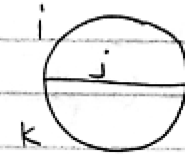
if (i,j,k) is not q-admissible.

ie) 

$\simeq$ 

For example, if (i,j,k) is not q-admissible then

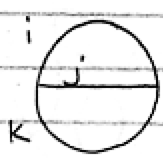


$=$ 

$= 0$

by prev. Thm.

If (i,j,k) is q-admissible, then

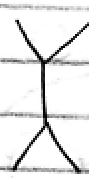
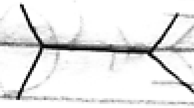


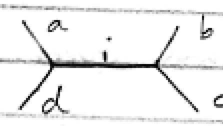
$\neq 0$ so 

$\simeq 0$.

proof of recoupling thm:

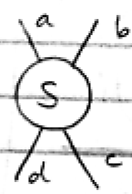
Want to show each spin network is a lin. comb of the other.

We get  $\simeq \sum \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}$ 

for unique $\begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}$ if 

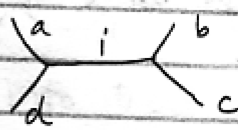
form

a basis of the v. space X where X is the space of linear combs. of spin networks of the form:



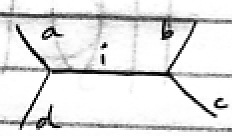
modulo Kauffman bracket relations and modulo $\simeq$ (equivalence).

Show linearly indep q_i span X.

Lemma 1: 

$((a,i,d),(b,c,i) \in AD_q)$

are linearly indep. vectors in X.

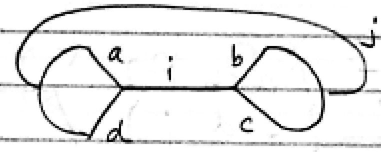
pf: Suppose $\sum_{\substack{i \text{ st} \\ (a,i,d),(b,c,i) \in AD_q}} \alpha_i$ 

$\simeq 0$

Show $\alpha_i = 0$.

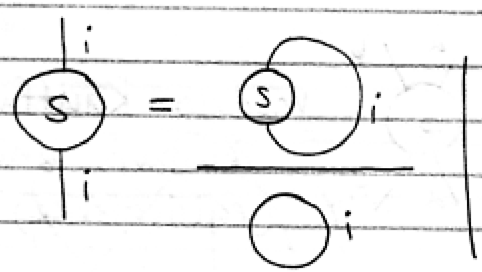
Take both sides of eqn & close it up so we get an eqn involving #'s.

So— by the defn of $\simeq$ we see

$\sum \alpha_i$  $\simeq 0$

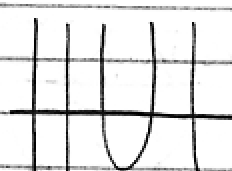
Recall from 1st quarter, Schur's Lemma —

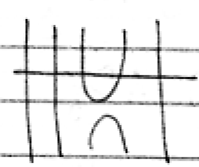
①  $= 0$ if $i \neq j$

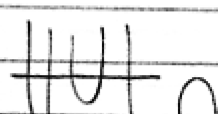
②  if i is admissible so $i\bigcirc \neq 0$.

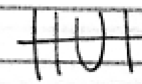
use these to look at eyeglasses above.

obtained by closing up $|i$ & LHS.

<u>Note</u>:  $= 0$ since

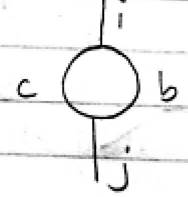
we proved  is zero.

by topology  $= 0$ and $\cap \neq 0$

$\Rightarrow$  $= 0$.

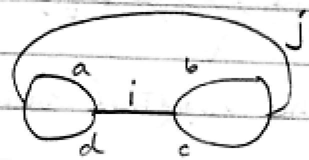
Recall T. Lieb alg — comb. of caps, cups, circles w/ no crossings.

If $i \neq j$ then



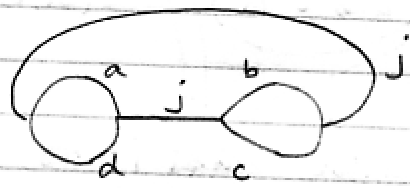
$= 0$

$\Rightarrow$ whole picture:



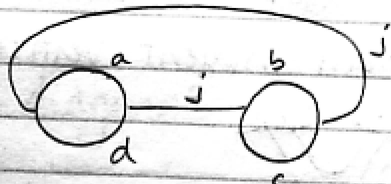
$= 0$

So we only need to consider when $\underline{\underline{i = j}}$.

α_j 

$= 0$

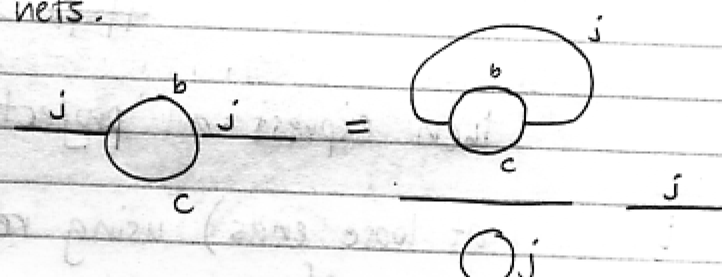
So we only need to show



$\neq 0$

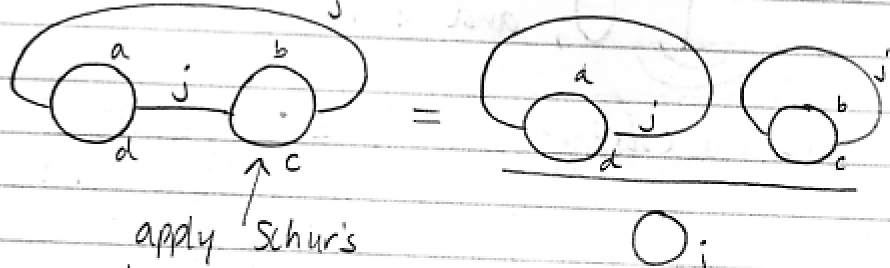
simplify this into theta nets.

By Schur [2], we get



if j is q-admissible.

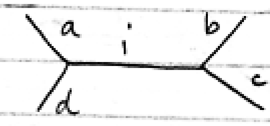
So —



apply Schur's lemma to RH lens

$\neq 0$ by prev. thm since top isn't zero.

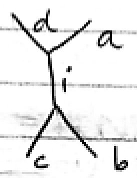
Thus $\alpha_j = 0$ so they're linearly indep.

<u>Lemma 2</u>:  span X

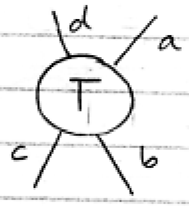
where

$((a,d,i), (b,c,i) \in AD_q)$.

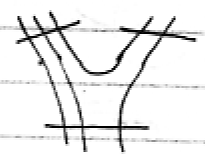
we'll prove it for the rotated version:

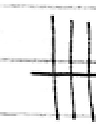
Show  span the corresponding space where

$(a,b,c,d) \longrightarrow (d,a,b,c)$

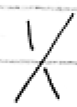
Given  we want it to be equivalent to a lin. comb. of these elements.

Take T, express all trivalent vertices using defn.

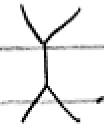


then express all projectors  (except those at loose ends) using recursion relation in terms of

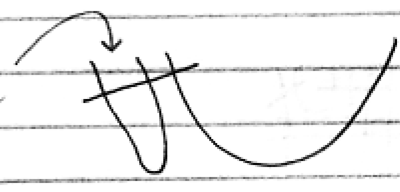
$|$, $\cup$, and $\cap$.

Eliminate all crossings via  $= A\,|| + A^{-1}\begin{smallmatrix}\cup\\ \cap\end{smallmatrix}$.

Get a lin. comb. of planar tangles built from spin-½ strands |, $\cup$ and $\cap$ (w/ no crossings)

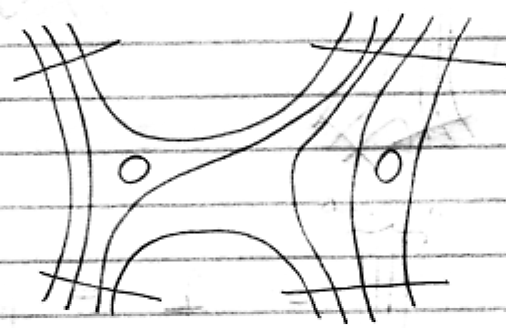
So – show any such planar tangle is a lin. comb of 

since

$\bigcirc$ = 

If strands leave where they entered, we get zero.

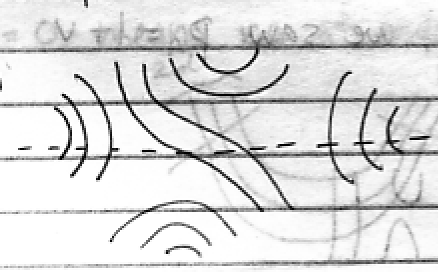
so consider:

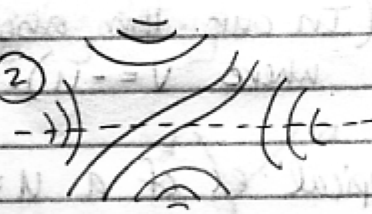


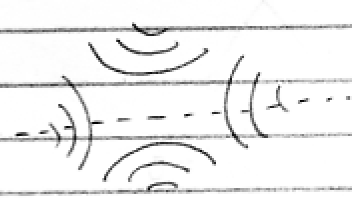
We can eliminate all loops via $\bigcirc = -(A^2 + A^{-2})$

Problem – the diagonal strands

The possibilities are:

① 

② 

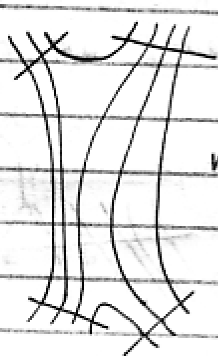
③ No diagonal strands: 

want to chop our cases in half:

Draw horizontal line crossing minimal number of strands:

- - - - -

say, n strands



n strands (go all the way through)

<u>Note:</u>

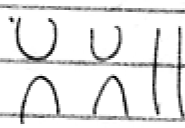
n-strands

$$= \underset{\underset{TL_n}{\cup\!\!\!\!\;\;}}{1} = \underset{\underset{TL_n}{\cup}}{p_n} + \underset{\underset{TL_n}{\cup}}{u}$$

(identity in temp lieb alg)

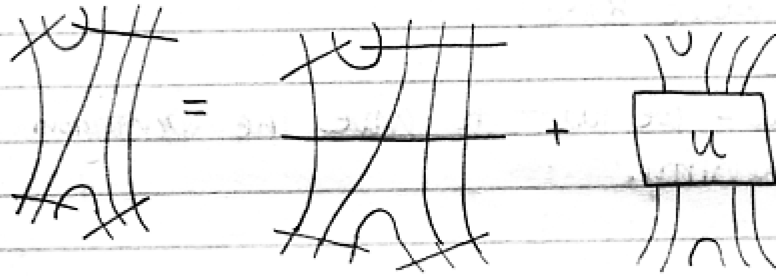
where u is a linear comb. of nontrivial products of $e_i \in TL_n$.
(In our thm about p_n, we saw $p_n = 1 + v$ where $v = -u$).

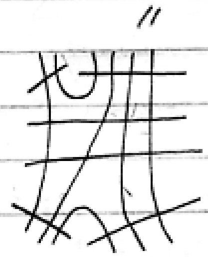
typical ex of a u =

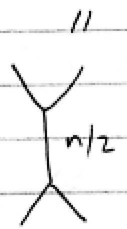


projector, so square is itself

So —



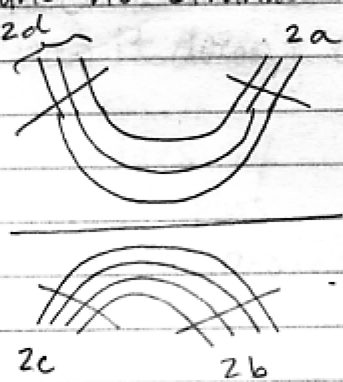


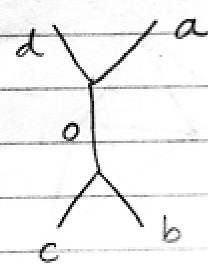


this is a linear comb. of planar tangles w/ < n strands going from top to bottom.

So: we've reduced the case of n-strands going from top to bottom to the case of < n.
(this is our inductive step)
By this induction, it suffices to handle the case n=0.

n=0 means no strands are going through





# of strands

0 = zero length.

$\Rightarrow$ (a=d, b=c by Δ ineq.)

So we're done!

Now – we want to prove the pentagon identity.