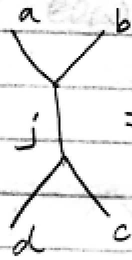
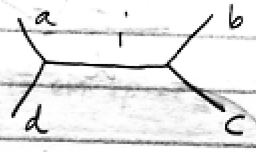
6/5/01

Last time we saw:

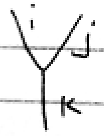
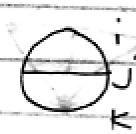


$$= \sum_{\substack{i \text{ st} \\ (i,b,c)\ \&\ (i,a,d) \in AD_q}} \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}_q$$



6j symbols allow us to write one of  in terms of the other.

where $(i,j,k) \in AD_q$ means  is well-defined

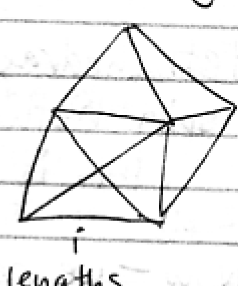
and  $\neq 0$, in fact  $\neq 0$.

In particular — $i,j,k \leq \frac{r-1}{2}$, i,j,k are q-admissible

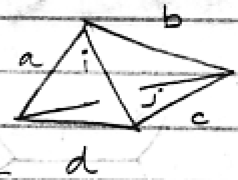
Let M be a compact oriented 3-manifold and let Δ be a triangulation.

Now — concoct a number $Z(M,\Delta)$ as follows: (we'll show it doesn't depend on the triangulation)

① Label all edges w/ q-admissible spins such that edges around any triangle are a q-admissible triple.



② For each tetrahedron calculate a <u>tet net</u>:

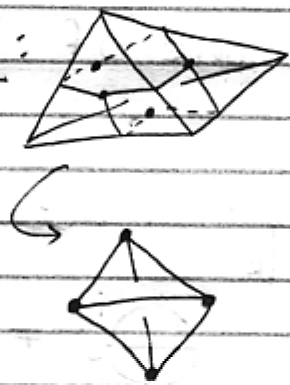


tetrahedron in triangulation

<u>Poincaré duality</u>:

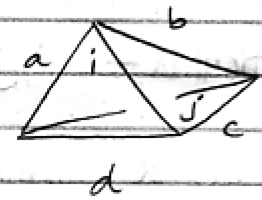
(*)

<u>Recall</u>:



Poincare dual of tetrahedron is tetrahedron

(*) Poincare dual of



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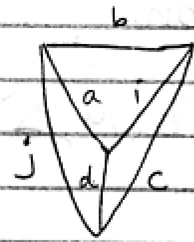


inherit edge labels from above — see which edges meet

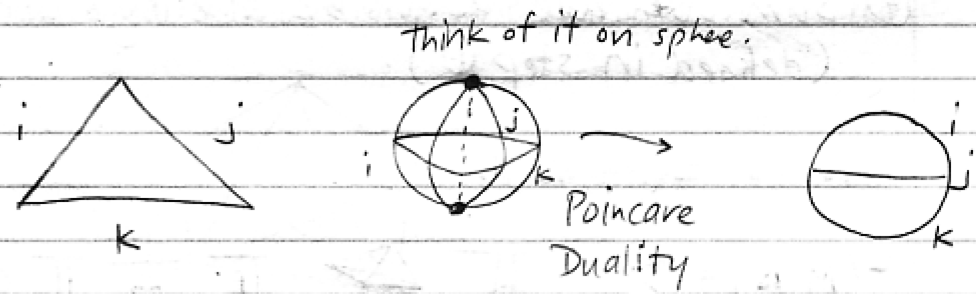
tetrahedral spin network

(this is what we mean for "calculate tet net")

Now — we get a number from the tetrahedral spin network.

Evaluating this tetrahedral spin network we get a number, called the <u>tet</u> <u>net</u>.

③ For each triangle, we calculate the reciprocal of a Theta net.

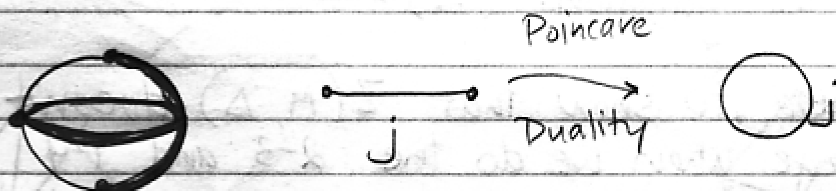


triangle in triangulation.

So we get a factor of $\dfrac{1}{\Theta^{i\,j}_{k}}$

④ For each edge we get a factor of $\bigcirc_j$.

$$\bigcirc_j = (-1)^{2j}\,[2j+1]_q$$



⑤ For each vertex, we get a factor of $\frac{1}{D}$ where

$$D = \sum_{\substack{q\text{-admissible} \\ \text{spins } j}} \bigcirc_j^{\,2}$$

"The dimension of the category of a q-admissible reps of $U_q\,sl(2)$."

$$= \sum_{\substack{q\text{-admissible} \\ j}} [2j+1]_q^2$$

(mentioned in Mueger's talk)

⑥ Multiply all the quantities in steps ② - ⑤
Take the product over tetrahedra, triangles,
edges of vertices of these numbers.

⑦ Sum over all q-admissible labellings
(defined in step ①).

This is $Z(M, \Delta)$.

$$Z(M,\Delta) = \sum_{\text{labellings}} \prod_{\text{tetrahedra}} \triangle \prod_{\text{triangles}} \frac{1}{\Theta} \prod_{\text{edges}} \bigcirc \prod_{\text{vertices}} \frac{1}{D}$$

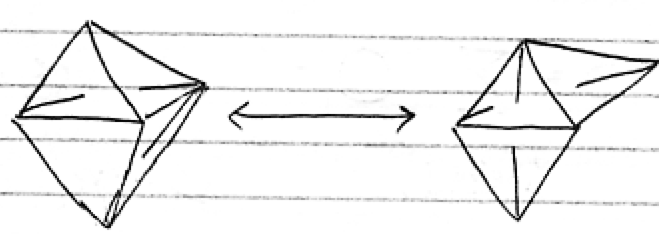
<u>Note</u>: $\frac{1}{\Theta}$ is never undefined since

$\Theta^{i}_{j\,k}$ is nonzero if (i, j, k) is q-admissible.

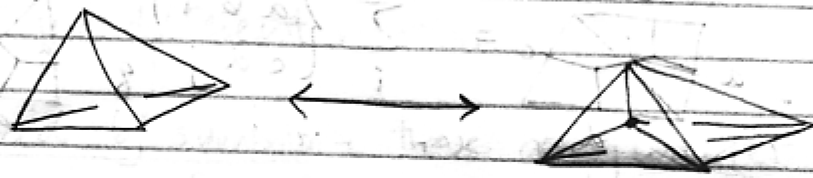
<u>Thm</u>: $Z(M, \Delta)$ doesn't depend on the triangulation. (Δ)

<u>proof</u>: We need to show that $Z(M, \Delta)$ doesn't change when we do the 2-3 and 1-4 Pachner moves.

<u>2-3</u>:

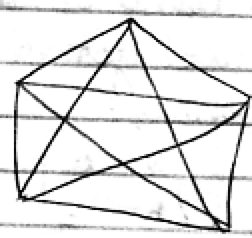


<u>1-4</u> :



We know how to show invariance under 2-3 move

<u>Note</u> :

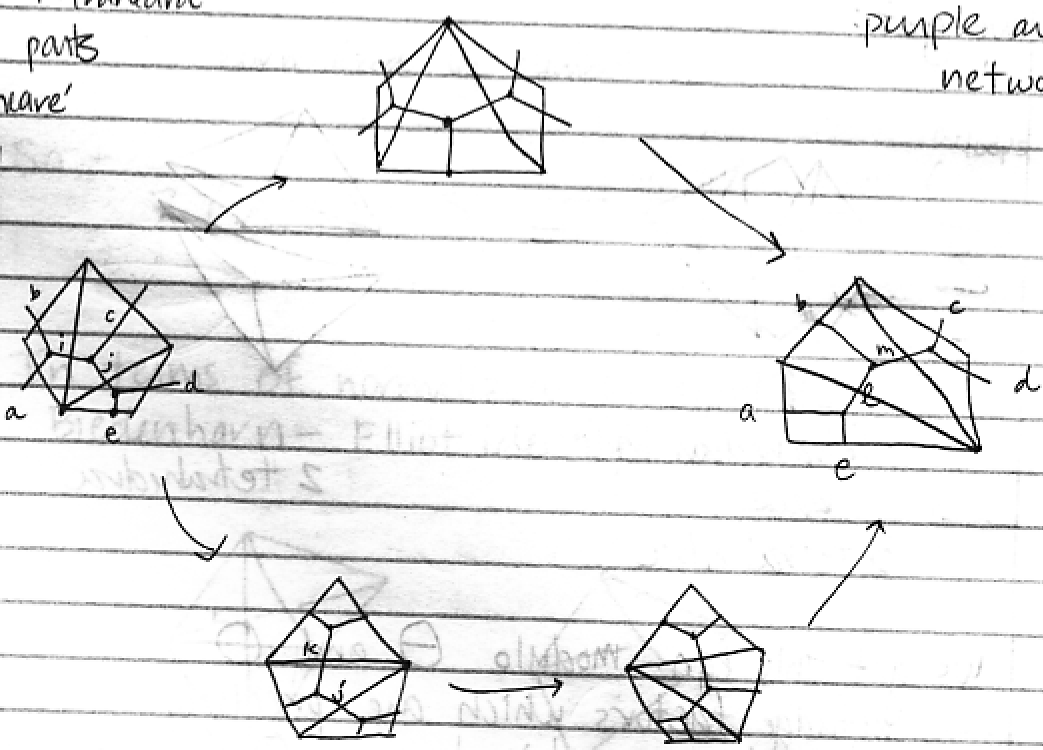


4-simplex

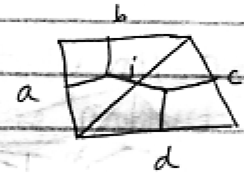
remove any vertex — we get a tetrahedron.
the 4-simplex has 5 tetrahedral faces.
The 2-3 and 1-4 moves come from chopping
4-simplex's boundary into 2 parts.

Chop up tetrahedra
into 3 parts
then Poincaré
duality

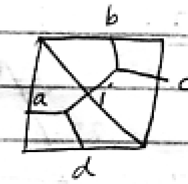
purple are spin
networks



So —



$$= \sum_i \begin{Bmatrix} a & b & i \\ c & d & j \end{Bmatrix}_q$$



Using 6j symbols to write the left-hand spin network as lin. comb of those on the right in 2 ways, we get Biedenharn - Elliot identity:

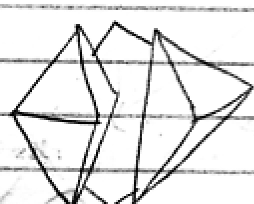
$$\sum_K \{ \quad \}_q \{ \quad \}_q \{ \quad \}_q = \{ \quad \}_q \{ \quad \}_q$$

The tet nets are related to 6j symbols. So — we can rewrite this eqn in terms of tet-nets!

We get:

see why this comes out from pentagons on prev pg.

$$\sum_K$$



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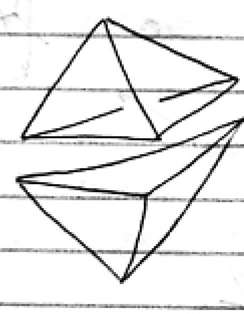


tetrahedron in back

3 tetrahedra

2 tetrahedra

top move — gives us 2 bottom way — gives us 3 tetrahedra

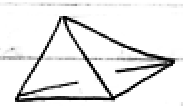
We get this eqn. modulo Θ and $\bigcirc$ fudge factors which are precisely one Θ for every triangle, one $\bigcirc$ for every edge.

Now – the 1-4 move.
We will use different notation:

(pg 3 handout) <u>Normalized 6j symbols</u> – these are just tet nets divided by four factors of $\frac{1}{\sqrt{\theta}}$ w/ θ's corresponding to four vertices in our tet net
ie) by 4 triangles in our original tetrahedron

We call this $\begin{bmatrix} a & b & i \\ c & d & j \end{bmatrix}_q$ and we get

$$\prod_{\text{tetrahedra}} \begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix} = \prod_{\text{tetrahedra}}$$



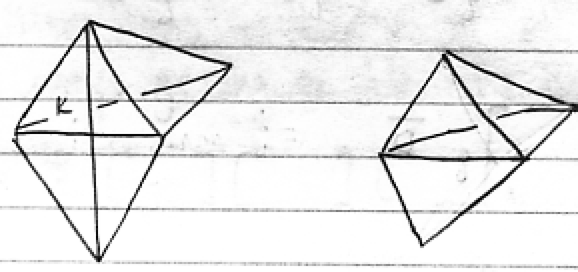
$$\prod_{\text{triangles}} \frac{1}{\theta}$$

normalized 6j symbol

So –

$$Z(M, \Delta) = \sum_{\text{labellings}} \prod_{\text{tet}} \begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix}_q \prod_{\text{edges}} \overset{\text{loops}}{\bigcirc} \prod_{\text{vertices}} \frac{1}{D}$$

In terms of normalized 6j symbols,
Biedenharn – Elliot identity:



is — (over)

$3 \longleftrightarrow 2$

$$\sum_k O_k [\quad]_q [\quad]_q [\quad]_q = [\quad]_q [\quad]_q$$

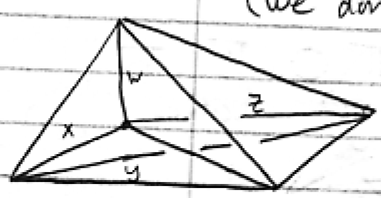
We'll need this and orthogonality identity:

$$\sum_j O_i O_j \begin{bmatrix} - & - & j \\ - & - & i \end{bmatrix}_q \begin{bmatrix} - & - & k \\ - & - & j \end{bmatrix}_q = \delta_{ik}$$

could just as well be k since sum is zero unless $i = k$.

We can get between any triangulations of M w/ the same number of vertices using only 2-3 and bubble moves, but we need 1-4 move to change number of vertices.

Sketch of proof that $Z(M, \Delta)$ is invariant under 1-4 move.
(we don't count things on both sides of the eqn.)

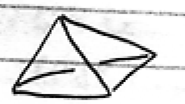


$$\sum_{w,x,y,z} O_w O_x O_y O_z [\quad]_q [\quad]_q [\quad]_q [\quad]_q \frac{1}{D}$$

by (BE identity) $$= \sum_{x,y,z} O_x O_y O_z [\quad]_q [\quad]_q [\quad]_q \frac{1}{D}$$

by orthogonality $$= \underbrace{\sum_{y,z} \frac{O\,O}{O}}_{=D} [\quad]_q \frac{1}{D}$$

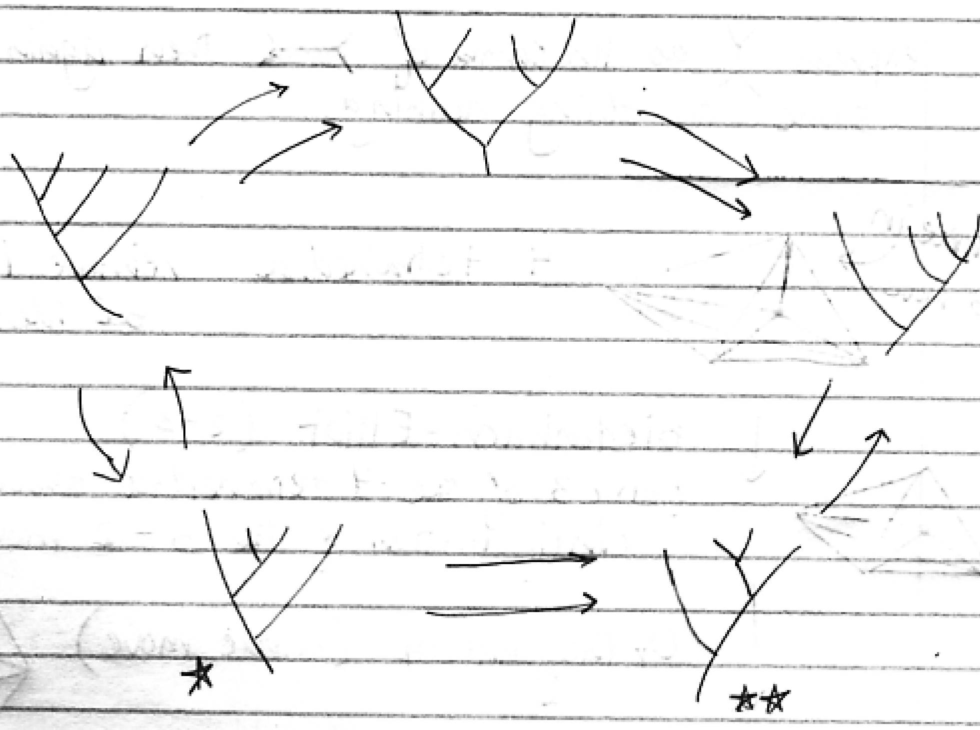
by magic identity $=$



$[\quad]_q$

Magic identity:

$$D = \frac{1}{O_j} \sum_{\substack{a,b \\ \text{st } (a,b,j) \\ q\text{-admissible}}} O_a O_b$$



to prove the 1-4 move, use →
pencil proves 2-3 move

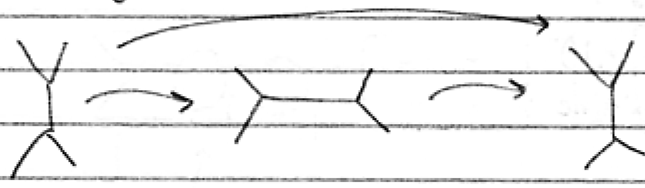
This picture lets us write ★ as a lin. comb in 2 ways getting an identity like

[ ] = Σ [ ][ ][ ][ ]

* This should somehow be related to the 1-4 move. *

How is this related to how we actually prove the 1-4 move?

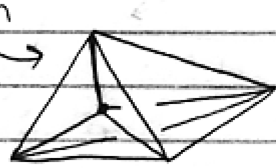
Orthogonality:



express Y as lin comb of >–< then again, same as doing nothing.

---

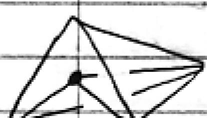
remove in 2-3 move



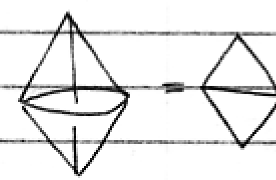
4 tetrahedra remove 1 via BE identity

↓ Biedenharn-Elliot (2-3)
(any 3 of the 4 tetrahedra above have a configuration of the 3 – 2 move)

3 tetra.



↓ Orthogonality (bubble move)



1 tetra.

↓ Magic Identity

1 tetra.