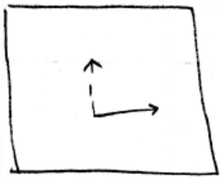
4/10/02

$$\begin{tikzcd} E \arrow[r, "\overset{t}{\sim}"] \arrow[d, "p"'] & B \times F \arrow[d] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

t = trivialization

Here tang. space is v. space

2 types of trivilizations: right & left handed



right handed — classified as # of twists as go up page + # of twists as go across page (takes 2 integers)

$$\begin{tikzcd} \pi_1(T^2) = & \mathbb{Z} \times \mathbb{Z} \arrow[d] \\ \pi_1(GL(2)) = & \mathbb{Z} \end{tikzcd}$$

same for left handed — again takes 2 integers to specify # twists in both ways

Want to classify trivializations up to homotopy.

very little of what we did last time was actually specific to the torus.

Recall:

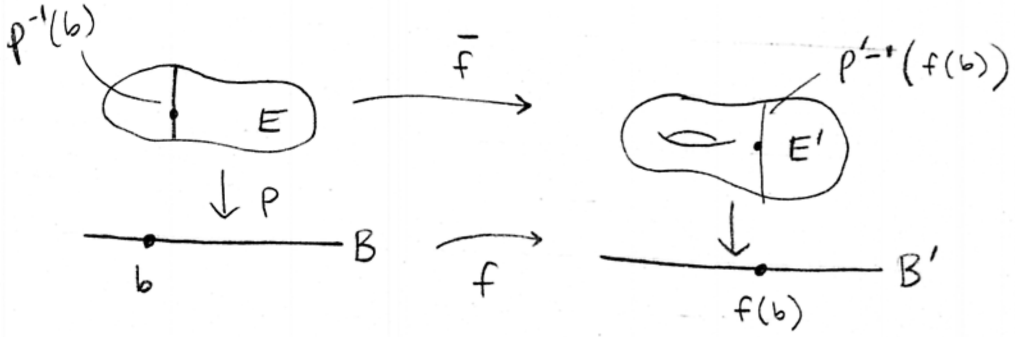
$$\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$$

bundle

morphism of bundles:

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

st square commutes



the fact that square commutes means that fiber over $b \in B$ is sent into fiber over $f(b) \in B'$ (not onto) via $\bar{f}$

Trivial Bundle: specific bundle where $E = B \times F$

$$\begin{tikzcd} B \times F \arrow[d, "\pi_1"] \\ B \end{tikzcd}$$

(projection onto 1st component)

Trivializable Bundle: A bundle isomorphic to a trivial bundle.

ie $\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$ st $\exists$ a trivialization: (isomorphism to trivial bundle)

$t =$ "trivialization"

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[d, "p"'] & B \times F \arrow[d] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

Started looking at classifying trivializations of a trivializable bundle (did for an example $B =$ torus) demanded linearity over each fiber.

Thm: Given a trivialization of the bundle

$$\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$$

, say

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[r, "\sim"'] \arrow[d, "p"'] & B \times F \arrow[d] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

we get other trivializations from automorphisms (of $B \times F$)

$$\begin{tikzcd} B \times F \arrow[r, "g"] \arrow[r, "\sim"'] \arrow[d] & B \times F \arrow[d] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

as follows – compose them:

$$\begin{tikzcd} E \arrow[r, "t"] \arrow[r, "\sim"'] \arrow[d, "p"'] \arrow[rr, bend left, "t'"] & B \times F \arrow[r, "g"] \arrow[r, "\sim"'] \arrow[d] & B \times F \arrow[d] \\ B \arrow[r, "1"] & B \arrow[r, "1"] & B \end{tikzcd}$$

so $t' = t \circ g$ is a new trivialization

Conversely, given any other trivialization, we get it this way from a unique automorphism of
$$\begin{tikzcd} B\times F \arrow[d] \\ B \end{tikzcd}$$
as follows:

$$\begin{tikzcd}
B\times F \arrow[d] \arrow[rr, bend left, "g = t^{-1}\circ t'"] & E \arrow[l, "t"'] \arrow[r, "t'"] \arrow[d] & B\times F \arrow[d] \\
B & B \arrow[l, "1"'] \arrow[r, "1"] & B
\end{tikzcd}$$

Show $t' = t\circ g$ where $g : B\times F \longrightarrow B\times F$

We let $g = t^{-1}\circ t'$ NOT $t'^{-1}\circ t$

(*)

So — to classify trivializations, we find one and classify automorphisms g.

$$\begin{aligned} g : B\times F &\longrightarrow B\times F \\ (b,f) &\longmapsto (b, \alpha_b(f)) \end{aligned}$$

g maps fibers to fibers

has to send fiber over b to something over b.

where $\alpha_b : F \xrightarrow{\sim} F$ is an iso, ie. diffeo of F

or $\alpha_{-} : B \longrightarrow \mathrm{Diff}(f)$

smooth map from B to Diffeos of f.

Last time $F = \mathbb{R}^2$ and we considered

$$\alpha : B \longrightarrow GL(2) \subseteq \text{Diff}(f)$$

We were "reducing the structure group" of our bundle from Diff(f) to GL(2).

E, B smooth manifolds, p smooth map
to what extent does our theorem/work depend on the fact that we're working in this category of manifolds?

We used Cartesian products! So this theory will also hold in any category w/ binary products.

(*) prev pg – up to here, we're working in any category w/ binary products

We can develop this bundle theory in any category w/ finite products:

- Diff – cat. of smooth manifolds, smooth maps
- Top – top. spaces & cont. maps
- Simp Set – simplicial sets & simplicial maps
- Chain Complexes – chain complexes & chain maps
- Affine Schemes – (commut. rings & ring homos)^{op}

Alg. Geometry: $x^2+y^2=1 \leftrightarrow k[x,y]/\{x^2+y^2-1=0\}$

Note - chain complex w/ one space = abel. grp.

Chain Complex:
$$\begin{tikzcd} 0 & C_0 \arrow[l] & C_1 \arrow[l, "d"'] & C_2 \arrow[l, "d"'] & \cdots \arrow[l] \end{tikzcd}$$
w/ $d^2 = 0$

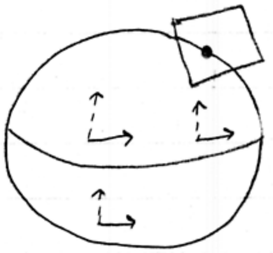
morphism bet. chain complexes:

$$\begin{tikzcd} 0 & C_0 \arrow[l] \arrow[d] & C_1 \arrow[l, "d"'] \arrow[d] & C_2 \arrow[l, "d"'] \arrow[d] & \cdots \arrow[l] \\ 0 & C_0' \arrow[l] & C_1' \arrow[l, "d'"] & C_2' \arrow[l, "d'"] & \cdots \arrow[l] \end{tikzcd}$$

w/ all squares commute

Not all interesting bundles are trivial or trivializable.

Ex) Tangent Bundle of S^2.



$$\begin{tikzcd} TS^2 \arrow[d, "p"] \\ S^2 \end{tikzcd}$$

A linear trivialization looks like this.

$$\begin{tikzcd} TS^2 \arrow[r, "t"] \arrow[d] & S^2 \times \mathbb{R}^2 \arrow[d] \\ S^2 \arrow[r] & S^2 \end{tikzcd}$$

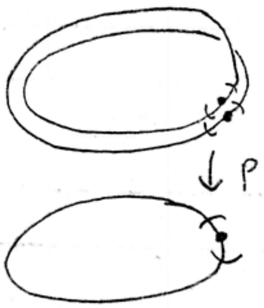
t linear on each fiber.

can't find a linear trivialization of
$$\begin{tikzcd} TS^2 \arrow[d, "p"] \\ S^2 \end{tikzcd}$$

since $\nexists$ a nowhere vanishing smooth v. field on S^2.
(or can't).

In fact - $\nexists$ any trivialization of
$$\begin{tikzcd} TS^2 \arrow[d] \\ S^2 \end{tikzcd}$$

Example : Mobius Strip bundle:



$$\begin{tikzcd} S^1 \arrow[d, "p"] \\ S^1 \end{tikzcd}$$

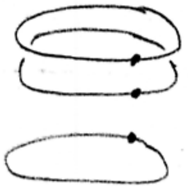
no isomorphism between these two

The fiber over any point is a pair of points

$\forall$ $b \in S^1$, the fiber over b is $\mathbb{Z}_2$.

* This bundle isn't trivial or trivializable.

The trivial bundle over S^1 w/ fiber $\mathbb{Z}_2$.



$$\begin{tikzcd} S^1 \times \mathbb{Z}_2 \arrow[d] \\ S^1 \end{tikzcd}$$

They can't be isomorphic — Mobius bundle has 1 component, $S^1 \times \mathbb{Z}_2$ has 2 components.

Since they're not isomorphic, the Mobius bundle isn't trivializable. However the Mobius bundle is locally trivializable (locally looks like $S^1 \times \mathbb{Z}_2$). Same for the sphere.

To talk about locally trivializable, we need to know what it means to talk about a bundle restricted to some open set.

Defn: Given a bundle $\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd}$ and an (open) set $U \subseteq B$

we define a bundle

$$\begin{tikzcd} E|_U \arrow[d, "p|"'] \\ U \end{tikzcd}$$

"E restricted to U"

(this is all pts in E over U)

where

$$E|_U = \{e \in E \mid p(e) \in U\}$$ and

$$p| : E|_U \longrightarrow U$$
$$e \longmapsto p(e)$$

We get

$$\begin{tikzcd}
E|_U \arrow[r, hook] \arrow[d, "p|"'] & E \arrow[d, "p"] \\
U \arrow[r, hook] & B
\end{tikzcd}$$

which commutes. This tells us we have a morphism of bundles.

This above square is a <u>pullback square</u>.

<u>Recall</u>:

$$\begin{tikzcd}
& \text{best answer} \arrow[dr, "\text{best}"] \arrow[dd, bend right=60, "\text{best}"'] & \\
& ? \arrow[u, "\exists!"'] \arrow[r, "?"] \arrow[d, "?"'] & B \arrow[d] \\
& A \arrow[r] & X
\end{tikzcd}$$

We can (in category theory) fill in these ? many ways.
The <u>best</u> way (universal) is called <u>pullback</u>
universal property of "best answer" are these maps which make everything commute.

above is universal property defining "pull back" given

$$\begin{tikzcd}
& B \arrow[d] \\
A \arrow[r] & X
\end{tikzcd}$$

In fact, $E|_{\mathcal{U}}$ is an example of a pullback.

So, if you're in a category w/ pullbacks we can talk about restrictions.

Defn: A bundle
$$\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$$
is locally trivializable

if $\forall b \in B$, $\exists$ open set $\mathcal{U} \ni b$ st

$$\begin{tikzcd} E|_{\mathcal{U}} \arrow[d, "p|"'] \\ \mathcal{U} \end{tikzcd}$$
is trivializable.

Want this to be true for a "cover" of B.

In other words, we need $E|_{\mathcal{U}_i}$ to be trivializable for some collection $\mathcal{U}_i$ of open sets that cover B.

(what does it mean to have a cover in a category?)

Note - If B compact, we can use finitely many $\mathcal{U}_i$'s.

Jim's comment: **Note** – Better to think of pulling back the whole "cover", not each piece of the cover & demand that those are trivializable.

Goal: We want to understand all locally trivializable bundles over B.

Suppose

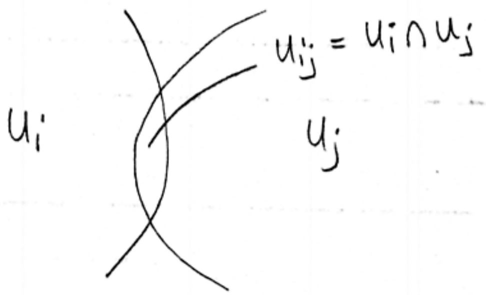
$$\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$$

locally trivializable and

pick an open cover U_i and trivialization t_i

$$\begin{tikzcd} U_i \times F_i \arrow[d] & E|_{U_i} \arrow[l, "t_i"'] \arrow[r, hook] \arrow[d, "p_i"] & E \arrow[d, "p"] \\ U_i & U_i \arrow[l, "1"'] \arrow[r, hook] & B \end{tikzcd}$$

What about when you have a pt. in intersection of 2 open sets?



trivialized over U_i, diff way over U_j

We can take $E|_{U_i}$ and $E|_{U_j}$ and restrict them further and we get $E|_{U_{ij}}$. So we get 2 trivializations of $E|_{U_{ij}}$, one being $t_i|_{E|U_{ij}}$ and other being $t_j|_{E|U_{ij}}$.

* This implies F_i is diffeomorphic to F_j *

$$\begin{tikzcd}
U_{ij} \times F_j \arrow[d] & E|U_{ij} \arrow[l, "t_j|_{E|U_{ij}}"', "\sim"] \arrow[r, "t_i|_{E|U_{ij}}", "\sim"'] \arrow[d] & U_{ij} \times F_i \arrow[d] \\
U_{ij} & U_{ij} \arrow[l] \arrow[r] & U_{ij}
\end{tikzcd}$$

so $F_i \cong F_j$ (diffeo) (fibers sent to fibers as before)

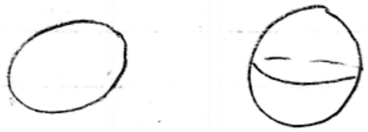
So if B is connected, we see all F_i's are diffeomorphic, so call them all F.

The fiber over any pt $\cong$ fiber over any other pt.

Defn: A fiber bundle is a locally trivializable bundle where all fibers are diffeomorphic.

Thm: A locally trivializable over a connected space is a fiber bundle.

Ex) non-connected



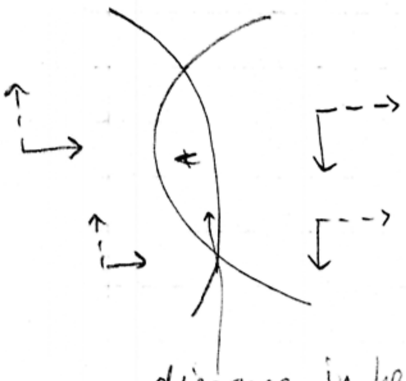
but different tang spaces.
$\mathbb{R}$ and $\mathbb{R}^2$.

So we get:

$$\begin{tikzcd}
U_{ij} \times F \arrow[d] & E|U_{ij} \arrow[l, "\sim"'] \arrow[r, "\sim"'] \arrow[d] & U_{ij} \times F \arrow[d] \arrow[ll, bend right, "g_{ij}"'] \\
U_{ij} & U_{ij} \arrow[l] \arrow[r] & U_{ij}
\end{tikzcd}$$

Prop: We get maps $g_{ij} = t_i^{-1} \circ t_j \Big|_{E|U_{ij}}$

the change of trivialization!



but g_{ij} allows us to have a diffeo between our 2 trivializations.

$\alpha_{ij} : U_{ij} \longrightarrow \mathrm{Diff}(F)$