4/16/02

- idea of cardinality of a manifold — Euler characteristic.

$$\chi(M+N) = \chi(M) + \chi(N)$$
$$\chi(M \times N) = \chi(M) \cdot \chi(N)$$

- Look at "flags" of v. spaces over $\mathbb{C}$

$$0 \subset \underset{\uparrow \text{ dim } 1}{V_1} \subset \underset{\nwarrow \text{ dim } 2}{V_2} \subset \dots \subset \underset{\nwarrow \text{ dim } n.}{V}$$

These form a "flag manifold," F_n

- Thm: $\chi(F_n) = n!$

- "$\mathbb{F}_1 = \mathbb{C}$" ($\mathbb{F}_1$ is field w/ 1 elt)
  $\chi(\mathbb{C}) = 1.$

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we're classifying fiber bundles...

We've seen that any fiber bundle

$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$

w/ fiber F can be described as follows:
(recall – fiber bundles are locally trivial by defn)

Pick an open cover U_i of B st

$$\begin{array}{c} E|_{U_i} \\ \downarrow \\ U_i \end{array}$$

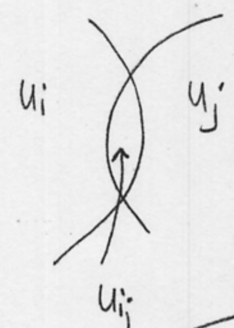
are trivializable.

(trivialization – iso. to a trivial bundle)

Pick trivializations:

$$\begin{tikzcd} E|_{U_i} \arrow[r, "t_i", "\sim"'] \arrow[d] & U_i \times F \arrow[d] \\ U_i \arrow[r, "1"] & U_i \end{tikzcd}$$

Let $U_{ij} = U_i \cap U_j$. We have 2 trivializations on the overlap.



$$\begin{tikzcd} U_{ij} \times F \arrow[d] & E|_{U_{ij}} \arrow[l, "t_j"'] \arrow[r, "t_i"] \arrow[d] & U_{ij} \times F \arrow[d] \arrow[ll, bend right, "t_i^{-1} \circ t_j = g_{ij}"'] \\ U_{ij} & U_{ij} \arrow[l, "1"'] \arrow[r, "1"] & U_{ij} \end{tikzcd}$$

We have:

$$g_{ij}(b,f) = (b,\ \alpha_{ij}(b)f)$$

since diagram commutes & b gets sent all the way around

where $\alpha_{ij}(b) : F \xrightarrow{\sim} F$ or

$$\alpha_{ij} : U_{ij} \longrightarrow \mathrm{Diff}(F)$$

At each pt in overlap we have a diffeo relating i^{th} to j^{th} picture.

<u>Defn</u>: If $G \subseteq \mathrm{Diff}(F)$ is a subgroup and

$\alpha_{ij} : U_{ij} \longrightarrow G \hookrightarrow \mathrm{Diff}(F)$ we say

$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$ is a <u>G-bundle</u> <u>w/ fiber F</u>.

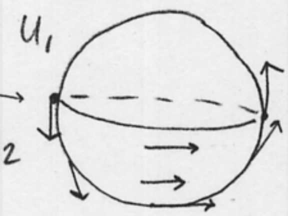
More generally – if G is any group w/ homo. to $\mathrm{Diff}(F)$:

$$\begin{array}{ccc} \alpha_{ij}: U_{ij} & \longrightarrow & G \\ & & \downarrow \\ & & \mathrm{Diff}(F) \end{array}$$

Example: Tangent bundle of S^2.

$$\begin{array}{c} TS^2 \\ \downarrow \\ S^2 \end{array}$$

see next pg (2)

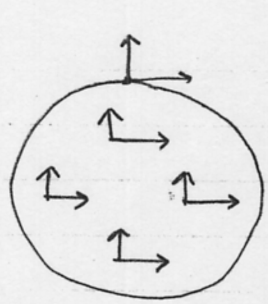


cover S^2 w/ 2 hemispheres U_1 = N. hemi + a little more
U_2 = S. hemi + more.

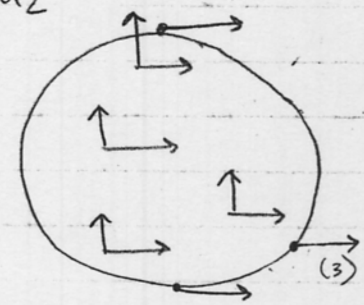
$U_1 \cap U_2$ =  band

$$\begin{array}{c} TS^2 \\ \downarrow \\ S^2 \end{array}$$ is a $GL(2)$ bundle w/ fiber $\mathbb{R}^2$.

U_1



U_2

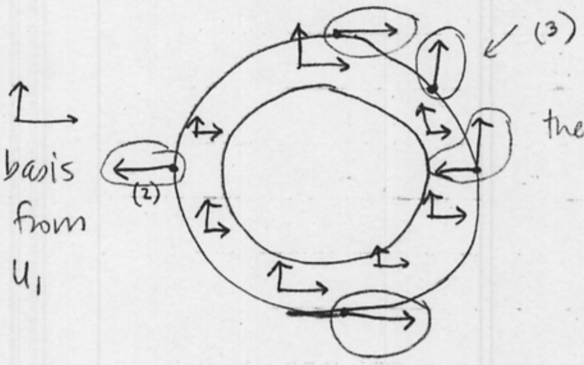


$$\begin{tikzcd} TS^2\big|_{U_1} \arrow[r, "t_1", "\sim"'] \arrow[d] & U_1 \times \mathbb{R}^2 \arrow[d] \arrow[l, bend right, "t_1^{-1}"'] \\ U_1 \arrow[r] & U_1 \end{tikzcd}$$

turn basis of $\mathbb{R}^2$ into basis of TS^2 via t_1^{-1}

$U_1 \cap U_2$: (annulus)

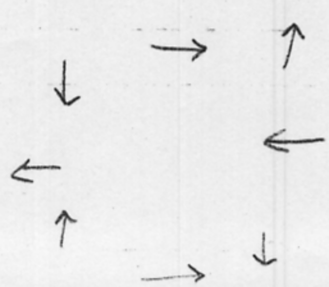
basis from U_1



these are the arrows from U_2

this is conformal inversion

the U_2 arrows do 2 complete rotations.



So here –

$$\alpha_{12} : \underbrace{U_1 \cap U_2}_{\text{annulus}} \longrightarrow GL(2)$$

which executes a 4π rotation as we go around the annulus.
We could build lots of $GL(2)$ bundles w/ fiber $\mathbb{R}^2$, one for each integer n by using α_{12}'s which execute a $2\pi n$ rotation.

If: $n=2$ we get the "tangent bundle"

$n=1$ we get a "spiner bundle"

In fact, any α_{12} gives

$$\pi_1(\alpha_{12}) : \pi_1(U_1 \cap U_2) \longrightarrow \pi_1(GL(2)) \quad \text{a homomorphism}$$

$$: \quad \mathbb{Z} \longrightarrow \mathbb{Z}$$
$$x \longmapsto n$$

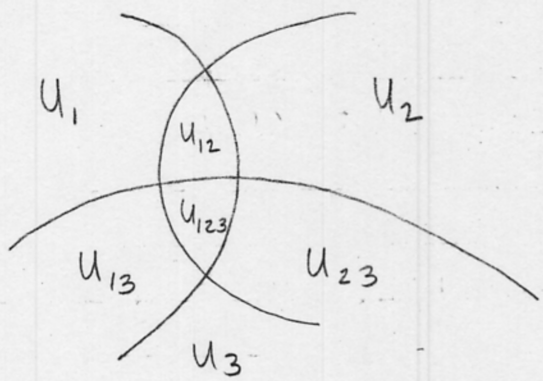
$GL(2)$ consists of $\det = -1$, $\det = +1$ parts

$$S^1 \approx SO(2) \approx GL_+(2) \subseteq GL(2)$$

<u>Note</u>: A homo of $\mathbb{Z}$ to $\mathbb{Z}$ is just a #.

So far we've focused on the case where $B = U_1 \cup U_2$ (only 2 sets in our open cover).

What about more?



In general, we get lots of functions

$\alpha_{ij} : U_{ij} \longrightarrow G$ but they'll satisfy a condition:

$$\alpha_{ij}\alpha_{jk}\alpha_{k\ell}\cdots\alpha_{yz} = \alpha_{iz} \quad \text{on } U_i \cap \ldots \cap U_z$$

recall:

$\alpha_{ij} = t_i^{-1} \circ t_j$, so above becomes

$$t_i^{-1} \circ \underbrace{t_j \circ t_j^{-1}} \circ \underbrace{t_k \circ t_k^{-1}} \circ t_\ell \cdots \circ t_y^{-1} \circ t_z = t_i^{-1} \circ t_z = \alpha_{iz}$$

This is just equivalent to:

$$\alpha_{ij}\alpha_{jk} = \alpha_{ik} \quad \forall\, i,j,k \quad \text{on } U_{ijk} = U_i \cap U_j \cap U_k$$

<u>AND</u> $\alpha_{ii} = 1 \quad \forall\, i$

Note – this follows since G is a group
$\alpha_{ii}\alpha_{ii} = \alpha_{ii} \Rightarrow \alpha_{ii} = 1$

(prev pg bottom) <u>Note</u>: This tells us how to do 2 changes or no changes.

Ex – These imply the rest:

$$\alpha_{ij}\alpha_{ji} = \alpha_{ii} = 1$$

<u>Defn</u>: Given a manifold B and a group G and an open cover U_i of B, we define a <u>Cech n-chain</u> to be a collection of functions

$$\gamma_{i_0 \cdots i_n} : U_{i_0} \cap \cdots \cap U_{i_n} \longrightarrow G$$

A <u>Cech 1-chain</u> is a <u>cocycle</u> if

$$\gamma_{ij}\gamma_{jk} = \gamma_{ik} \qquad \text{and} \qquad \gamma_{ii} = 1$$

(equation between 2-chains)

<u>Thm</u>: Suppose $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ is a G-bundle with fiber F, U_i is an open cover of B. We can find $\alpha_{ij} : U_i \cap U_j \longrightarrow G$, a Cech 1-cocycle,

s.t.

$$\begin{tikzcd} E \arrow[r, "\sim"] \arrow[d] & \left(\coprod_i U_i \times F\right)/\sim \arrow[d] \\ B \arrow[r, "\sim"] & B \end{tikzcd}$$

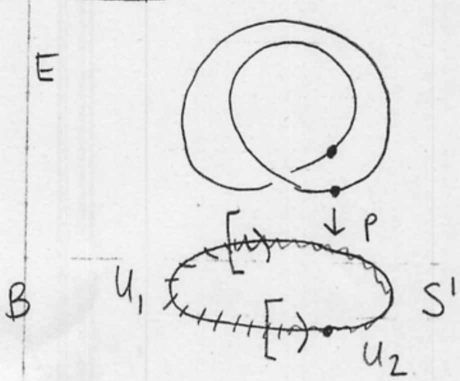
↓ disj't union

where $\left(\coprod_i U_i \times F\right)/\sim$

where $$\underset{\in\, U_i \times F}{(b,f)} \sim \underset{\in\, U_j \times F}{(b',f')}$$

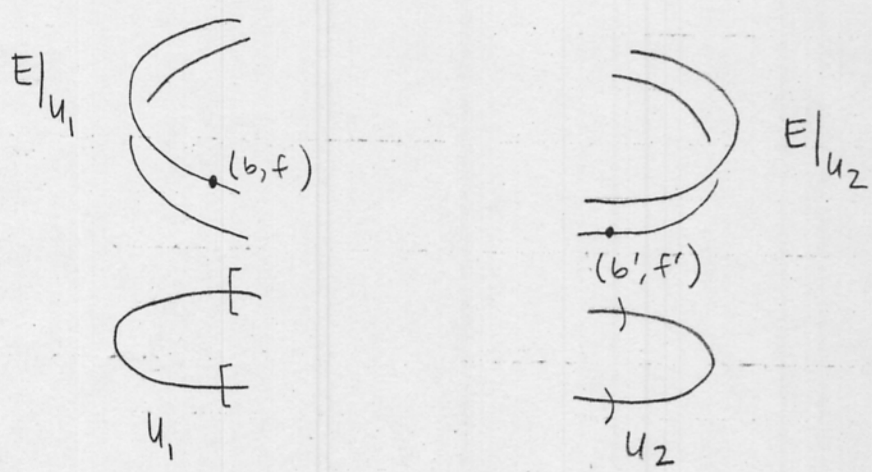
if $b' = b$ and $f' = \alpha_{ij}(b)(f)$.

<u>Example</u>:



locally trivial

over U_1 we have trivial bundle: (same for U_2)



We call this "clutching"

glue together these 2 trivial bundles w/ twist.
We build the Mobius bundle by sticking these together.