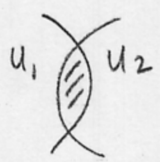
$$\begin{array}{c} U_i \times F \\ \downarrow \\ U_i \end{array}$$



glue together bundles over overlap.

4/18/02

always takes values in subgroup

Thm: Given a G-bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ with fiber F $(G \subseteq \mathrm{Diff}(F))$

we can choose an open cover U_i of B st $E|_{U_i}$ are trivializable,

$$\begin{tikzcd}
E|_{U_i} \arrow[r, "\overset{\text{trivializ.}}{\sim}", "t_i"'] \arrow[d] & U_i \times F \text{ (trivial bundle)} \arrow[d] \\
U_i \arrow[r, "1"] & U_i
\end{tikzcd}$$

and picking trivializations t_i we get transition functions:

$$\begin{tikzcd}
(U_i \cap U_j) \times F \arrow[d] & E|_{U_i \cap U_j} \arrow[l, "t_j"'] \arrow[r, "t_i"] \arrow[d] & (U_i \cap U_j) \times F \arrow[d] \arrow[ll, bend right, "g_{ij}"'] \\
U_i \cap U_j \arrow[r, "1"] & U_i \cap U_j \arrow[r, "1"] & U_i \cap U_j
\end{tikzcd}$$

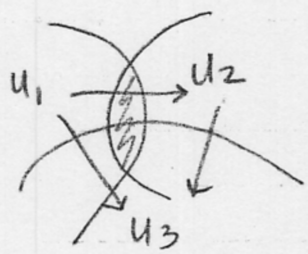
$g_{ij} = t_i \circ t_j^{-1}$ (transition function)

where $g_{ij} : (\underset{U_i \cap U_j}{\overset{\cap}{b}}, f) \longmapsto (b, \alpha_{ij}(b)(f))$ and

$\alpha_{ij} : U_i \cap U_j \longrightarrow G$

α_{ij}'s satisfy conditions:

(ie. α_{ij} form a Cech 1-cocycle)



$$\alpha_{ij}\,\alpha_{jk} = \alpha_{ik} \quad \text{on } U_i \cap U_j \cap U_k$$

and we can reconstruct our bundle from the α_{ij}'s.

<u>Converse</u>: Given α_{ij}'s construct a G-bundle so that these are the transition functions.

More precisely:

<u>Thm</u>: Given a manifold B, a manifold F, $G \subseteq \mathrm{Diff}(F)$, an open cover U_i of B and a Cech 1-cocycle

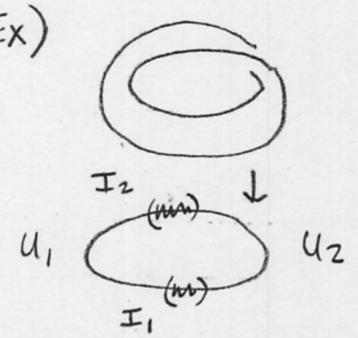
$$\alpha_{ij} : U_i \cap U_j \longrightarrow G$$

there is a G-bundle w/ fiber F with α_{ij} as its Cech 1-cocyle (as in prev. thm).

<u>proof</u>: Let's build $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$.

Continued next pg...

Ex)

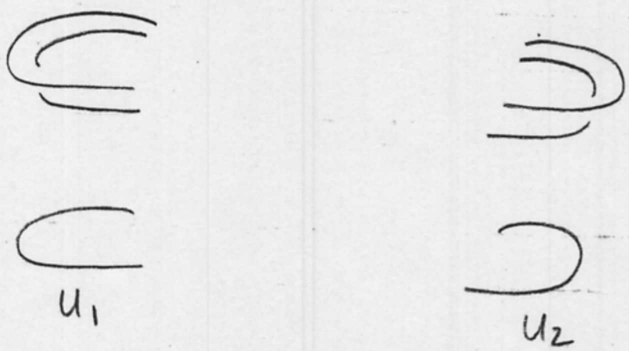


this is just the edge of Mobius strip.

How do we build the mobius strip bundle?

Given 2 open sets, we need to know how to glue open sets together.

$U_1 \times \mathbb{Z}_2$



$U_2 \times \mathbb{Z}_2$

$F = \mathbb{Z}_2$

$\alpha_{12} : U_1 \cap U_2 \longrightarrow \mathbb{Z}_2$

$G = \mathrm{Diff}(F) \cong \mathbb{Z}_2$

$\mathbb{Z}_2$ acts as diffeos of itself by left mult.

$U_1 \cap U_2 = I_1 \cup I_2$

How do we define α_{12}?

$\alpha_{12}\big|_{I_1} = 1$ (ident) $\quad\quad$ $\alpha_{12}\big|_{I_2} = -1$ (flip & get twist)

<u>proof</u>: let's build
$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$

We form
$$\begin{array}{c} \coprod_i U_i \times F \\ \downarrow \\ \coprod_i U_i \end{array}$$

(trivial bundle) and then mod out by the equivalence relation

$$\underset{U_i \times F}{(b,f)} \sim \underset{U_j \times F}{\left(b, \alpha_{ij}(b)(f)\right)} \qquad \text{where } b \in U_i \cap U_j$$

to get
$$\begin{array}{c} E = \coprod_i U_i \times F / \sim \\ \downarrow \\ B = \coprod_i U_i / \sim \end{array}$$

What about the transition functions?

If $b \in U_i \cap U_j \cap U_k$ then
$$\begin{aligned} \underset{U_i \times F}{(b,f)} &\sim \left(b, \alpha_{ij}(b) f\right) \\ &\sim \left(b, \alpha_{jk}(b)\alpha_{ij}(b) f\right) \\ &\sim \underset{U_i \times F}{\left(b, \alpha_{ki}(b)\alpha_{jk}(b)\alpha_{ij}(b) f\right)} \end{aligned}$$

so we get in trouble unless $\alpha_{ij}\alpha_{jk}\alpha_{ki} = 1$

The above is why we need the Cech cocycle condition.

We've checked given a bundle, get a cocycle, then go back & get the same bundle.

Quest: Now — when do 2 different open covers or Cech cocycles give isomorphic G-bundles.

First - fix open cover U_i of B, fix F and $G \subseteq \mathrm{Diff}(F)$, fix G-bundle w/ fiber F

$$\begin{array}{c} E \\ \downarrow \\ B \end{array}$$

Different local trivializations give different Cech 1-cocycles. How different?

t's are another trivialization

$$\begin{tikzcd}
U_j \times F & & U_i \times F \arrow[ll, bend right, "g_{ij}"'] \\
& E|_{U_i \cap U_j} \arrow[ul, "t_j"'] \arrow[ur, "t_i"] \arrow[dl, "t'_j"] \arrow[dr, "t'_i"'] & \\
U_j \times F \arrow[uu, "h_j"] & & U_i \times F \arrow[uu, "h_i"'] \arrow[ll, bend left, "g'_{ij}"]
\end{tikzcd}$$

top & bottom commute

$$g_{ij} = t_i^{-1} \circ t_j$$

we want to write g_{ij}' in terms of g_{ij}

$$1_{E|_{U_i \cap U_j}} = t_i \circ g_{ij} \circ t_j^{-1} = t_i \circ t_i^{-1} t_j t_j^{-1}$$

$$g_{ij}' = t_i'^{-1} t_j'$$

So $\boxed{g_{ij}' = t_i'^{-1} \circ t_i \circ g_{ij} \circ t_j^{-1} \circ t_j'}$

In this case, we say g_{ij}' and g_{ij} differ by a "coboundary" - they give isomorphic G-bundles.

let's call $h_i = t_i'^{-1} \circ t_i$; $h_j = t_j'^{-1} \circ t_j$

we get $\boxed{g_{ij}' = h_i g_{ij} h_j^{-1}}$

We define Cech n-chains to be collections of smooth functions

$$K_{i_0 \cdots i_n} : U_{i_0} \cap \cdots \cap U_{i_n} \longrightarrow G$$

and we say 2 Cech 1-chains g_{ij} and g_{ij}' <u>differ</u> <u>by</u> <u>a</u> <u>coboundary</u> if for some Cech 1-chain h_i, we have

or are "cohomologous"

$$g_{ij}' = h_i g_{ij} h_j^{-1}$$

* Moral: Cohomologous Cech 1-chains yield isomorphic bundles.

<u>Review</u>: We say an open cover $\mathcal{U} = \{U_i\}$ is <u>finer</u> than $\mathcal{V} = \{V_j\}$ if $\forall i$, $\exists j$ st

$$U_i \subseteq V_j.$$

How do we turn a Cech cocycle for "coarse" cover:

$$\alpha_{ij} : V_i \cap V_j \longrightarrow G$$

into a cocycle

$$\beta_{k\ell} : U_k \cap U_\ell \longrightarrow G \ ?$$

Note – could have a problem if U_k sits inside more than one V_i.

Really – we should say a morphism

$$f : \mathcal{U} \longrightarrow \mathcal{V}$$

is a map sending each i to $f(i)$ s.t.

$$U_i \subseteq V_{f(i)}$$

and this gives for each cocycle $\alpha_{jj'} : V_j \cap V_{j'} \to G$
a cocycle

$$f^*(\alpha)_{ii'} = \alpha_{f(i)f(i')}\Big|_{U_i \cap U_{i'}}$$

If we let $H^1(B, \mathcal{U}, G)$ be Čech 1-cocycles mod coboundaries we get

$$f^* : \underbrace{H^1(B, \mathcal{V}, G)}_{\text{coarse}} \longrightarrow H^1(B, \underbrace{\mathcal{U}}_{\text{finer}}, G)$$

Check: 1) this trick carries cocycles to cocycles
2) carries cohomologous ones to cohomologous ones.

In fact, we can form a "direct limit" or "colimit"

$$H^1(B, G) = \varinjlim_{\mathcal{U}} H^1(B, \mathcal{U}, G)$$

Want to describe <u>all</u> G-bundles. Making cover finer & finer, we can describe more & more G-bundles.

<u>Thm</u>: The set of isomorphism classes of G-bundles over B is in 1-1 correspondence with $H^1(B, G)$.

Ideas behind proof:

1) Any specific G-bundle corresponds to $\alpha \in H^1(B, \mathcal{U}, G)$ for a sufficiently fine cover $\mathcal{U}$.

2) Given 2 covers $\mathcal{U}$, $\mathcal{U}'$ we can find a cover finer than both (take $\cap$ of open sets in $\mathcal{U}$ w/ $\mathcal{U}'$).

$$\begin{tikzcd}
\mathcal{U} \arrow[dr, "f"] & \\
& \mathcal{V} \\
\mathcal{U}' \arrow[ur, "f'"'] &
\end{tikzcd}$$