4/23/02

Examples of fiber bundles:

Thm: The G-bundles are classified, up to isomorphism, by $\check{H}^1(B,G)$ = 1^{st} Cech cohomology.

Examples:

① Given any manifold M, TM is a $GL(n)$-bundle with fiber $\mathbb{R}^n$.

a) If $M = S^2$, TM is not trivializable.

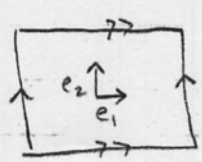
trivializable means isomorphic to a trivial bundle.

If it were,

$$\begin{tikzcd} TS^2 \arrow[r, "\sim"] \arrow[d] & S^2 \times \mathbb{R}^2 \arrow[d] \\ S^2 \arrow[r, "\sim"] & S^2 \end{tikzcd}$$

Then the standard basis would give 2 everywhere lin. indp. v. fields on S^2.
But - there's not one - hairy ball thm.

b) If $M = T^2$, we saw TM is trivializable (in lots of ways):



c) If $M =$ the 2-holed torus 

TM is not trivializable due to:

<u>Thm</u>: A compact manifold M has a nowhere vanishing smooth vector field iff $\chi(M) = 0$.

Idea: chop M up into convex (or contractible) polytopes:

count vertices, edges, faces, etc.
Then:

$$\chi(M) = V - E + F - \dots$$

χ is independent of the way we chop up M into polytopes.

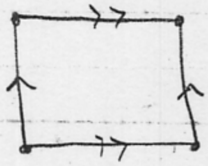
Ex) 

3 vertices
3 edges
2 faces

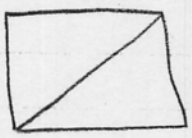
$$\chi(S^2) = 2 \neq 0$$

$S^2 =$ 2 triangles

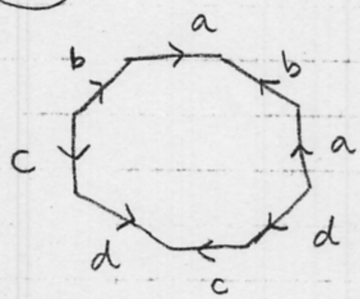
puff up face

Ex)  = 

vertices = 1 (all identified)
edges = 2
faces = 1 $\qquad \chi(T) = 1 - 2 + 1 = 0$

or $T^2 =$  $\qquad V - E + F = 1 - 3 + 2 = 0$

Ex)  relation in π_1 = $aba^{-1}b^{-1}cdc^{-1}d^{-1}$

use an octagon: 

$V = 1$
$E = 4$
$F = 1$

$\chi($$) = 1 - 4 + 1 = -2 \neq 0$

So there is no nowhere vanishing smooth vector field on 

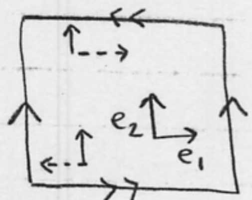
Thus – the tangent bundle of the 2-holed torus is <u>not</u> trivializable.

d) If M is a g-holed torus,

$\chi(M) = 2-2g$ so TM is trivializable only for the torus.

② <u>Thm</u>: Every compact orientable 3-manifold M has TM trivializable.

Klein bottle:



take out bottom,
it comes out at top looking like

For any manifold, TM is a $GL(n)$ bundle; it's <u>orientable</u> if we can reduce it to a $GL_0(n)$ bundle where

$$GL_0(n) = \{x \in GL(n) \mid \det x > 0\}$$

ie – we can pick a globally well-defined notion of "right-handed" basis of $T_x M$.

If TM (or any G-bundle) were trivializable, we could reduce the group down to <u>trivial</u> group.

So, if TM is trivializable, then M is orientable.

a) $M = S^3$, TM is trivializable

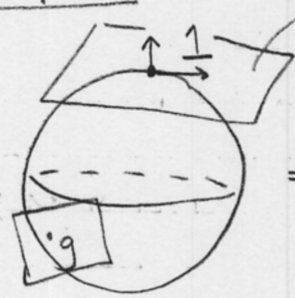
$$S^3 \cong SU(2) \cong \{\text{unit quatenions}\}$$
$$SU(2) = \{2\times 2 \text{ unitary matrices w/ } \det = 1\}$$

So S^3 is a Lie group.

<u>Thm</u>: If G is a Lie group then TG is trivializable.

<u>sketch of proof</u>:

tangent space at identity $= su(2)$ lie alg. $\cong \mathbb{R}^3$



$= G$

$1 \in G$ (ident elt)

Left mult. by a group elt

$$L_g : G \longrightarrow G, \quad x \longmapsto gx$$

$$dL_g : T_1 G \xrightarrow{\sim} T_g G$$

takes basis from tang. space @ 1 to basis at $T_g G$.

dL_g takes a basis of $T_1 G \approx \mathbb{R}^n$ into a trivialization of TG.

$S^1 \times S^1$ = the torus; since S^1 is a grp, so $S^1 \times S^1$ also is

b) The torus is also a Lie grp, so above thm also shows $T(T^2)$ is trivializable.

③ In higher dimensions, "most" manifolds have nontrivial tangent bundles.

④ Let's start w/ a group G, and find all G-bundles over M (some manifold).

Let $G = U(1)$. What are all the $U(1)$-bundles (up to iso) over M like?

Here, S^1 = {unit complex #'s}
$\simeq U(1)$ = {1×1 unitary matrices}
= $\mathbb{C}$ #'s w/ abs. value 1

Note – unit $\mathbb{R}$eal #'s = $\{-1, 1\} = \mathbb{Z}_2$

Fact: Of all spheres, only S^0 = {unit real #'s} $= \mathbb{Z}_2$,

S^1 = {unit complex #'s} $= U(1)$,

S^2 = {unit quaterions} $= SU(2)$

can be made into Lie groups;

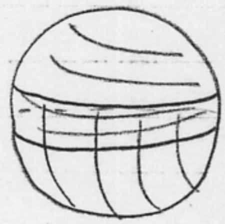
S^7 = {unit octonions} (not a grp – not assoc)

is the only other sphere w/ trivializable tangent bundle.

We want to understand $U(1)$-bundles over any M.

a) Take $M = S^2$. Want to build a $U(1)$ bundle over it.

U_1



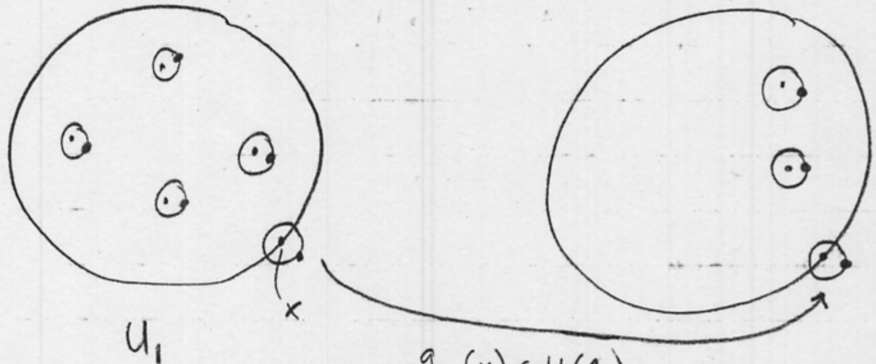
U_2

chop into hemispheres w/ overlap.

Cover S^2 w/ U_1, U_2, put trivial $U(1)$ bundles over these open sets, form a $U(1)$ bundle over S^2 using transition funct.

$$g_{12} : U_1 \cap U_2 \to U(1).$$

iden. elt of $U(1)$ denoted by • on circle.



g_{12} is a function from the overlap — the band around the equator, so the $U(1)$ bundles are classified (up to isomorphism) by an integer, the winding number of the map

$$g_{12} : U_1 \cap U_2 \longrightarrow U(1)$$

see: http://www.math.ucr.edu/home/baez/braids.html

<u>Note</u>: $H^1(X) = H^1(X; \mathbb{Z})$ coefficients from $\mathbb{Z}$, but could use any abel. grp.

<u>Thm</u>: If G is a discrete abelian group, then

$$\check{H}^1(M, G) \cong H^1(M, G)$$

where $H^1(M, G)$ is the ordinary cohomology of M w/ coefficients in G.

But $U(1)$ (as a Lie group) is not discrete, so how does this help?

<u>Thm</u>: If you have an exact sequence of abelian groups:

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \xrightarrow{\alpha} 0$$

$\operatorname{im} g = \ker \alpha$ but $\ker \alpha = A_3$ so g is onto

onto

$\operatorname{im} f = \ker g$

$\operatorname{im} 0 = \ker f \Rightarrow f = 1\text{-}1$

ie) $A_3 = A_2 / A_1$

then we get a "long" exact sequence of cohomology grps:

$$\ldots \check{H}^{n-1}(M, A_3) \xrightarrow{\partial} \check{H}^n(M, A_1) \longrightarrow \check{H}^n(M, A_2) \longrightarrow \check{H}^n(M, A_3) \xrightarrow{\partial} \check{H}^{n+1}(M, A_1) \ldots$$

(repeats every 3 terms, raise/lower power of n)

where – image of each homo is kernel of next one.

Let's apply this thm to:

$$\begin{array}{c} 0 \longrightarrow \mathbb{Z} \xhookrightarrow[\text{inc}]{} \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0 \\ \hphantom{0 \longrightarrow \mathbb{Z} \xhookrightarrow[\text{inc}]{} \mathbb{R} \longrightarrow} \cong \hphantom{\longrightarrow 0} \\ \hphantom{0 \longrightarrow \mathbb{Z} \xhookrightarrow[\text{inc}]{} \mathbb{R} \longrightarrow} U(1) \hphantom{\longrightarrow 0} \end{array}$$

We also need:

<u>Thm</u>: $\check{H}^n(M, \mathbb{R}) = 0$ (since $\mathbb{R}$ is contractible)
$(n>0)$

We get:

$$\begin{tikzcd} & 0 \arrow[d, phantom, "\shortparallel"] & \\ \check{H}^1(M, \mathbb{Z}) \arrow[r] & \check{H}^1(M, \mathbb{R}) \arrow[r] & \check{H}^1(M, U(1)) \arrow[dll, "\delta"'] \\ \check{H}^2(M, \mathbb{Z}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \check{H}^2(M, \mathbb{R}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & \cdots \\ H^2(M, \mathbb{Z}) & 0 & \end{tikzcd}$$

since $\mathbb{Z}$ is discrete

$$0 \xrightarrow{\alpha} \check{H}^1(M, U(1)) \xrightarrow{\delta} H^2(M, \mathbb{Z}) \xrightarrow{\beta} 0$$

$\operatorname{im} \alpha = \ker \delta$

to $\ker \delta = 0 \Rightarrow \delta$ is 1-1

$\Rightarrow$ δ is an iso.

$\operatorname{im} \delta = \ker \beta = H^2(M, \mathbb{Z})$ so δ onto.

So, since d is an iso:

$$\check{H}^1(M, U(1)) \cong H^2(M, \mathbb{Z})$$

So we get:

Thm: $U(1)$-bundles over M are classified by $H^2(M, \mathbb{Z})$.

Example: If M is a compact orientable 2-manifold (g-holed torus $g \geq 0$),

$$H^2(M, \mathbb{Z}) = \mathbb{Z}$$

(confirming what we guessed for $M = S^2$).

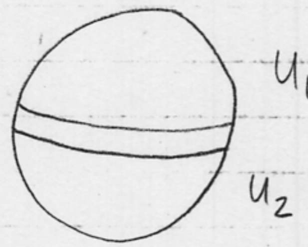
Example: $H^2(S^3, \mathbb{Z}) = 0$ so $U(1)$-bundles over S^3 are trivializable.

6) What about $SU(2)$ bundles?
This is a lot harder since $SU(2)$ is not abelian.

Thm: If M is a compact orientable 2-manifold, all $SU(2)$ bundles over M are trivializable.

what about $SU(2)$ bundles over S^3?

$$S^3 = U_1 \cup U_2$$



This gives us

$SU(2)$ bundles from transition functs

$$g_{12} : U_1 \cap U_2 \longrightarrow SU(2)$$

$\simeq$ homotopy equiv.

$$S^2$$

so we need to understand homotopy classes of maps from S^2 to $SU(2)$, so

$$\pi_2(SU(2))$$

Classifies $SU(2)$ bundles over S^3.

<u>Thm</u>: If G is compact, Lie group $\Rightarrow \pi_2(G) = 0$.

So: all $SU(2)$-bundles over S^3 are trivial!

In general - G-bundles over S^n are classified by homotopy classes of maps

$$g_{12} : S^{n-1} \longrightarrow G$$

i.e. $\pi_{n-1}(G)$

<u>Thm</u>: $\pi_3(SU(n)) = \mathbb{Z}$.

So – for example, $SU(2)$ bundles over S^4 are classified by an integer, <u>2nd Chern class</u>, or <u>instanton number</u>.