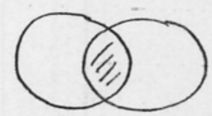
4/25/02

<u>Thm</u>: Given a manifold F and group $G \subseteq \mathrm{Diff}(F)$ then G-bundles over B w/ fiber F are classified, up to isomorphism by

$$[g_{ij}] \in \check{H}^1(B, G).$$

<u>Note</u>: · F appears in the hypothesis but not conclusion!

· The thm is true, so it means that classification doesn't depend on F.



$U_i \times F$ $\quad$ $U_j \times F$

So let's create F with $G \subseteq \mathrm{Diff}(F)$ in some "god-given" way starting from F.

· G is a Lie Group, hence a manifold, so choose $F = G$.

How do we get $G \subseteq \mathrm{Diff}(G)$?

Note— groups act on themselves by left, right mult or conjugation. But we can't use conjugation if G is abelian because we get a trivial action.

We'll use the left action of G on itself:

$$\begin{aligned} G &\hookrightarrow \mathrm{Diff}(G) \\ g &\longmapsto L_g \quad \text{where} \end{aligned}$$

$$\begin{aligned} L_g : G &\longrightarrow G \\ x &\longmapsto gx \end{aligned} \quad \text{is given by } \forall x \in G.$$

* This is a way to talk about bundles where $F = G$.

<u>Defn</u>: A G-bundle w/ fiber G (where $G \subseteq \mathrm{Diff}(G)$ by left mult) is called a <u>principal bundle</u>, $\begin{matrix} P \\ \downarrow \\ B \end{matrix}$

<u>Joke</u> – not "principle" since a "principle" bundle has moral fiber.

<u>Example</u>: The Möbius strip bundle $\begin{matrix} P \\ \downarrow \\ S' \end{matrix}$ has fiber $\mathbb{Z}_2$ acting on itself by multiplication.

<u>Thm</u>: Given a principal G-bundle $\begin{matrix} P \\ \downarrow \\ B \end{matrix}$ there exists, for any space F, w/ $G \subseteq \mathrm{Diff}(F)$, a G-bundle $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$ w/ fiber F classified by same elt. of $\check{H}^1(B, G)$.

we call $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ the bundle w/ fiber F associated to $\begin{array}{c} P \\ \downarrow \\ B \end{array}$.

Conversely, given any G-bundle w/ fiber F, it comes from a unique (up to iso) principal G-bundle.

proof: Given $\begin{array}{c} P \\ \downarrow \\ B \end{array}$, we get $[g_{ij}] \in \check{H}^1(B,G)$ and use this to construct $\begin{array}{c} E \\ \downarrow \\ B \end{array}$. Converse works the same way.

* Knowing how to classify principal bundles, we know how to classify them all.

Note: Often people consider G-bundles where we have a homo. $G \longrightarrow \text{Diff}(F)$ that's not necessarily 1-1 (so $G \nsubseteq \text{Diff}(F)$).
In this case, every G-bundle is described by some $[g_{ij}] \in \check{H}^1(B,G)$ but not necessarily a unique one, so we lose uniqueness in above Thm - but rest still works.

Example: Suppose V is a v. space and $G \subseteq GL(V)$ (invertible linear transf. of V) $\subseteq \text{Diff}(V)$

Given a principal G-bundle, $\begin{array}{c} P \\ \downarrow \\ B \end{array}$, the theorem gives us a G-bundle w/ fiber V.
These are called vector bundles. (Fiber is v. space)

If $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is a vector bundle, then the fiber $p^{-1}(b)$ over b is a v. space ($\cong V$).

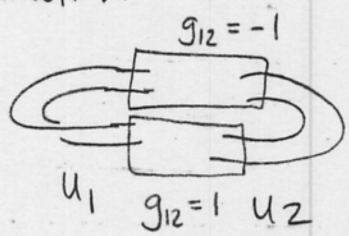
(transition functions act as linear transfs)

<u>Sub-example</u>: let $\begin{array}{c} P \\ \downarrow \\ B \end{array}$ be the Möbius strip bundle so $G = \mathbb{Z}_2$.

We want a v.space & subgrp of Diff(V).

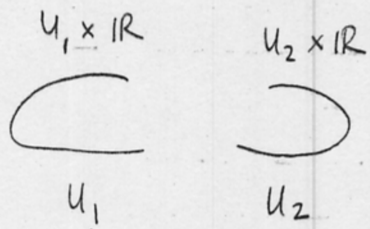
Take $V = \mathbb{R}$ and $\mathbb{Z}_2 = \{+1, -1\} \subseteq GL(\mathbb{R}) \cong \mathbb{R}$

We get a vector bundle from above construction.

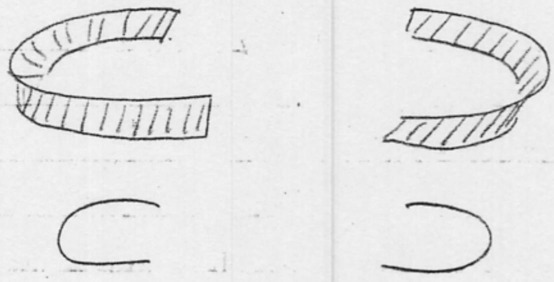


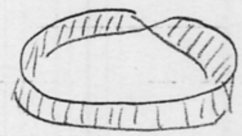
stick on transition functions on overlap.

$$g_{12} : U_1 \cap U_2 \longrightarrow \mathbb{Z}_2$$



Starting from:



and "clutching" we get 

*(Möbius strip bundle before was just the edge)

the honest to goodness Möbius strip (<u>vector bundle</u>)

Given a G-bundle w/ fiber F, say $\begin{matrix} E \\ \downarrow \\ B \end{matrix}$, our theorem gives us a principal G-bundle $\begin{matrix} P \\ \downarrow \\ B \end{matrix}$

with same $[g_{ij}] \in \check{H}^1(B, G)$. what's this like?

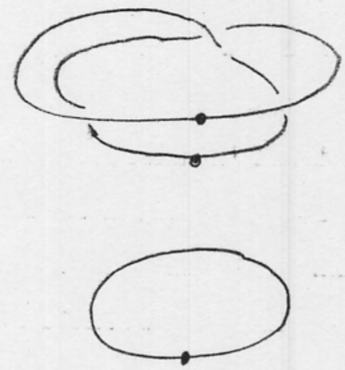
(Meaning what are the fibers over B like?)
They're somewhat like G.

What is the fiber over $b \in B$? We know it is diffeomorphic to G, but not in any god-given way.

<u>Note</u>: If $\begin{matrix} P \\ \downarrow \\ B \end{matrix}$ is a principal G-bundle, there's a right action of G on P which maps each fiber to itself.

$$\left.\begin{array}{l} g(hk) = (gh)k \\ L_g R_k = R_k L_g \end{array}\right\}$$ The assoc. law is a special case of commut. law

<u>Example:</u>



$\mathbb{Z}_2$

$$\begin{array}{c} P \\ \downarrow \\ B \end{array}$$

$\mathbb{Z}_2$ acts on P to "switch points in each fiber"

($1 = id$ does nothing
-1 switches the 2 pts)

In general — if we have a trivial principal G-bundle:

$$\begin{array}{c} P = B \times G \\ \downarrow \\ B \end{array}$$

Here G acts on P by $(b,h)g = (b, hg)$ $\forall\, (b,h) \in B \times G$.

If P is built from trivial G-bundles

$$\begin{array}{c} U_i \times G \\ \downarrow \\ U_i \end{array}$$

where U_i is an open cover using transition functions

$$g_{ij} : (U_i \cap U_j) \times G \longrightarrow (U_i \cap U_j) \times G$$

Can we get G to act on the right on P in a consistent way?

Need to check that given $(b,h) \in (U_i \cap U_j) \times G$ we get

$$g_{ij}\left[(b,h)g\right] = \left[g_{ij}(b,h)\right]g$$

* g_{ij}'s act as left mult by grp elt (since principal bundle)

Note: $g_{ij}(b,h) = (b, \alpha_{ij}(b)h)$ $\quad \alpha_{ij}(b) \in G$

so all we need is

$$(b, \alpha_{ij}(b)(h,g)) = (b, (\alpha_{ij}(b)h)g)$$

which is true since mult in grp is associative (or left & right mult commute)

Most people define a <u>principal G-bundle</u> to be a bundle

$$\begin{array}{c} P \\ \downarrow \\ B \end{array}$$

with a right action of G preserving fibers which is a locally trivializable G-bundle and G acts freely and transitively on each fiber.

<u>Question:</u> So: given a G-bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$ w/ fiber F,

$G \subseteq \mathrm{Diff}(F)$, what is the corresponding principal G-bundle $\begin{array}{c} P \\ \downarrow \\ B \end{array}$ like?

What is the fiber of P over $b \in B$, P_b?

<u>Answer:</u> The fiber of P over b is the space of "frames at b" where a <u>frame</u> is a map

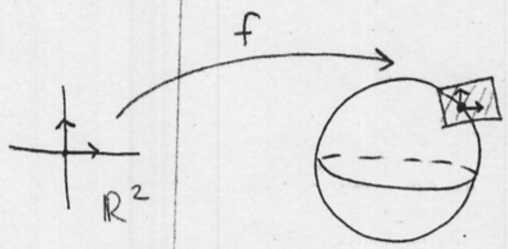
$$f : F \xrightarrow{\sim} E_b \qquad \text{(a diffeo)}$$

where E_b is the fiber of E over b st

$$f(x)g = f(xg).$$

Example: $\begin{array}{c} E = TS^2 \\ \downarrow \\ S^2 \end{array}$ has fiber $F = \mathbb{R}^2$

$G = GL(2)$



A <u>frame</u> is a map

$$\mathbb{R}^2 \xrightarrow{\sim} T_b S^2 \quad \text{st}$$

our condition $f(x)g = f(xg)$ says that our map f is linear.

So a <u>frame</u> is a basis of $T_b S^2$.