4/30/02 Connections on principal bundles

(book talks about connections on vector bundles)
· all forces of nature are connections
(ex- vector potential)

All forces of nature are described by connections:

• Maxwell's eqns (connection = vector potential, really 1-form A on spacetime
*good for electromagnetism $F = dA$
$*d*F = J$) (A is a connection)

• Yang-Mills eqns (describe weak & strong nuclear forces)

here - a connection is roughly a 1-form A w/ values in some Lie algebra.
For EM, the Lie alg. was $\mathbb{R} = \mathfrak{u}(1)$
(Lie alg of $U(1)$)

$\mathfrak{su}(2)$ gives the weak force
$\mathfrak{su}(3)$ gives the strong force

$$F = dA + \tfrac{1}{2}[A, A]$$

For $\mathbb{R}$, $[\,,\,] = 0$ (abelian) so doesn't show up

$$*d*F + *[A, *F] = J$$

· Einstein's eqns (describes gravity)

use Lie alg $\mathfrak{so}(3,1)$.

we'll focus on principal G-bundles and define connections on those;
the Lie algebras associated to the different forces of physics are really the Lie algebras of different choices of G "the gauge group."

Recall: For any Čech 1-cocycle g_{ij} on our "base" manifold B:

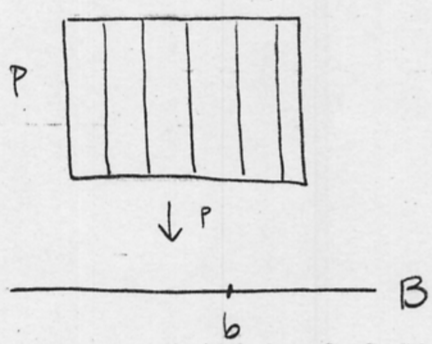
$$g_{ij} : U_i \cap U_j \longrightarrow G \quad \text{we get G-bundles,}$$

in particular, a <u>principal</u> G-bundle, $\begin{matrix} P \\ \downarrow p \\ B \end{matrix}$, ie a bundle where fiber is G and we think of $G \subset \mathrm{Diff}(G)$ via left translations:

$$g \longmapsto L_g \qquad L_g \in \mathrm{Diff}(G)$$

$$L_g(x) = gx \quad \forall x \in G.$$

<u>Example</u>: Let $G = \mathbb{R}$



The fibers $p^{-1}(b)$ all look like $\mathbb{R}$ (diffeo to $\mathbb{R}$) but aren't isomorphic in any "god-given" way.

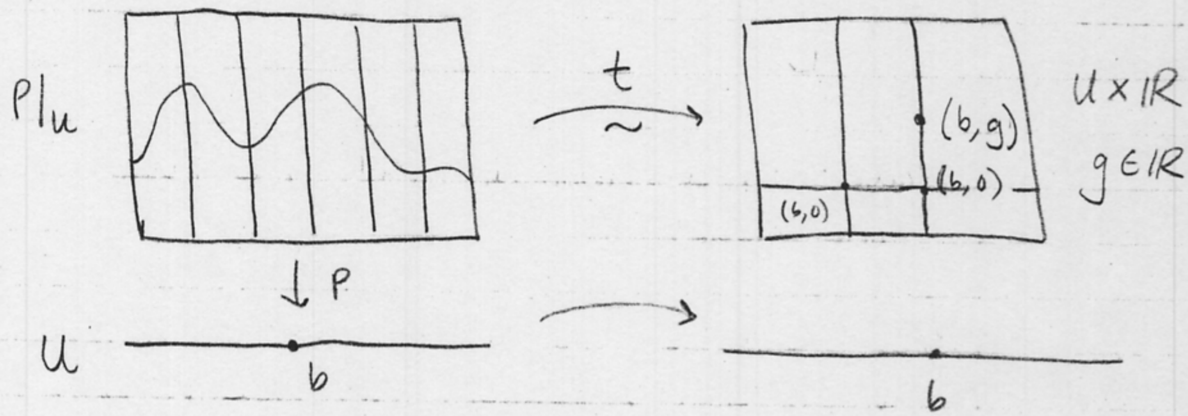
The point: * * The fibers $p^{-1}(b)$ $(b \in B)$ are diffeo to $\mathbb{R}$, but not in any god-given BEST way.

We know $0 \in \mathbb{R}$, but we can't tell on the fiber $p^{-1}(b)$ where 0 is.

We get a diffeomorphism

$p^{-1}(b) \xrightarrow{\sim} \mathbb{R}$ if we pick a trivialization of P.

In fact, it suffices to pick a local trivialization over $U \subseteq B$ where U is an open set and $b \in U$.



Trivial Bundle

Picking a trivialization means a commut. diagram, w/ trivial $\mathbb{R}$ bundle.

* here we know how to find zero on each fiber $p^{-1}(b)$. It's where $g=0$

the trivialization t is a diffeo, so we can look at inverse image of "zero" points over in $P|_U$. So, we call "zero" in $P|_U$ something that gets sent to $(b,0)$ via t.

Example: let B = our universe (some 4-manifold)
TB = tangent bundle
so fibers, are like
$F = \mathbb{R}^4$
We have no diffeo bet. F & $\mathbb{R}^4$ in a "best" way.

FB = frame bundle - a principal $GL(4)$-bundle
(4x4 invert. matrices)

frame - basis of vectors

$$\begin{array}{c} FB \\ \downarrow p \\ B \end{array} \qquad p^{-1}(b) = \left\{ \begin{array}{l} \text{all bases of the tangent} \\ \text{space } T_b B \text{ at } b \end{array} \right\}$$

(recall - for a principal bundle, the fibers look like the group).

Here - the fibers $F_b B = p^{-1}(b)$ are all diffeomorphic to $G = GL(4)$, since if we pick a basis of $T_b B$, every other basis can be written out as a matrix w/r/t the chosen one, so we get

$$F_b B \xrightarrow{\sim} GL(4)$$

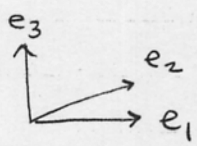
But as before, this diffeo

$$F_b B \xrightarrow{\sim} GL(4)$$

is not god-given (we made a choice).

Theorem: If B is any n-dim'l manifold its frame bundle
$$\begin{array}{c} FB \\ \downarrow \\ B \end{array}$$
is a principal $GL(n)$ bundle.

$e \in F_b B$

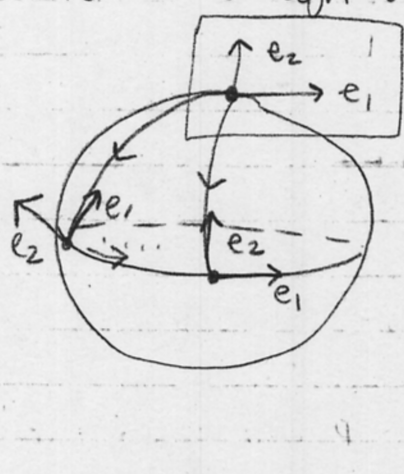


$T_b B$

Is this pointing in x direction?

To answer the question, we want to pick up our basis, bring over to b', and then answer quest. But this requires carrying our basis over w/out rotating it, which we can't do since spacetime is curved. This is the defn of curvature.

Ex)



but go down side, then right don't agree

Given a frame $e \in F_b B$ and a point $b' \in B$, how can we "carry" e over to b'?

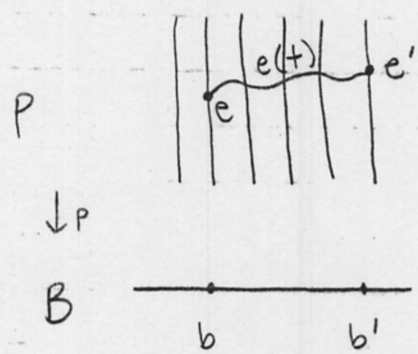
The answer could depend on the path from b to b'.

It also depends on the "connection" – a geometrical structure on the frame bundle
$$\begin{array}{c} FB \\ \downarrow \\ B \end{array}.$$

More generally – we can define a connection on a principal G-bundle?

$$\begin{array}{c} P \\ \downarrow p \\ B \end{array}$$

Ex) here, $G = \mathbb{R}$



$e \in p^{-1}(b)$

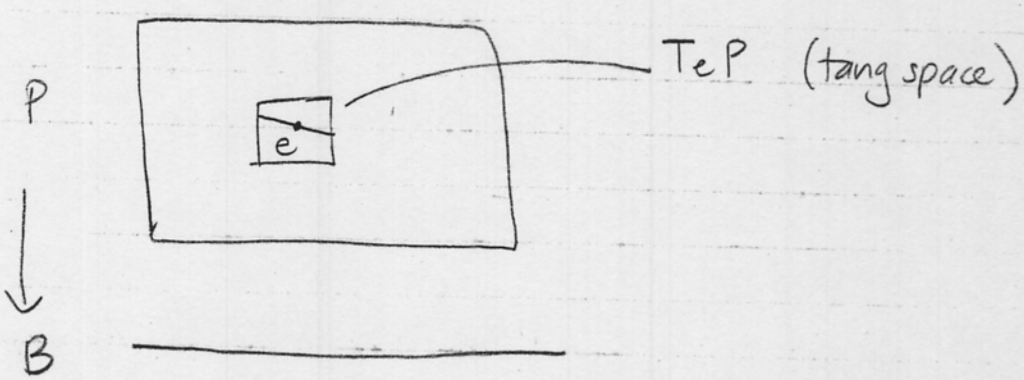
Want to take e to e'.
So, choose a path from b to b'.

If this were a trivial bundle, we could carry e horizontally across.

We'd like to carry $e \in p^{-1}(b)$ to $e' \in p^{-1}(b')$ w/ out twisting it around. A connection allows us to do this.

We want to carry e to e' st the curve $e(t)$ in P is "horizontal" – but we don't know what this means if we don't have a god-given way of comparing points in any fiber $p^{-1}(b)$ w/ points in G.

A <u>connection</u> will be a choice of "horizontal."



We want to pick a "horizontal subspace"

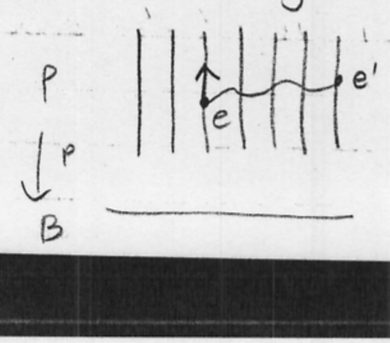
$$H_e \subseteq T_e P$$

A tangent vector in here will be declared to be horizontal.

If we choose a tang. vector $\uparrow$ to be "horizontal", we see that

dp sends this tang. vector to zero, so it wasn't "horizontal".

(differential) sends tang. vectors to tang. vectors



Not okay!

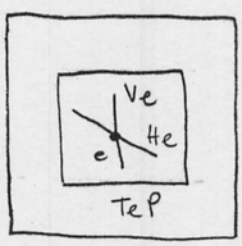
We'll figure out some conditions that the spaces H_e need to satisfy.

① A vector $0 \neq v \in H_e$ should not be <u>vertical</u>:
we say $v \in T_e P$ is vertical if
$dp(v) = 0$.

$$\begin{array}{c} T_e P \\ \downarrow dp \\ T_b B \end{array} \qquad p(e) = b.$$

<u>Note</u>: we can define $V_e \subseteq T_e P$ the <u>vertical</u> subspace without any choice of extra structure:
just define
$$V_e = \{ v \in T_e P \mid dp(v) = 0 \}$$
$$= \ker dp$$

But choosing H_e really involves a choice.



In short, we want $H_e \cap V_e = \{0\}$, and

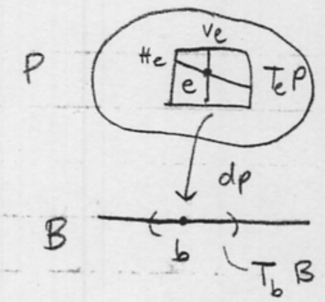
$$T_e P = H_e \oplus V_e$$ (meaning every elt of $T_e P$ is a pair of something horiz, something vert)

* We can decompose a tang vector into a horiz & vert. part.

Since we can decompose any tang. vector in $T_e P$ into a horiz, vert. part, we can look at tangent vectors & want vertical part to be zero. (This gets us horiz vectors)

If $T_e P \cong H_e \oplus V_e$ then consider:

$$dp : T_e P \longrightarrow T_b B$$



dp kills everything in V_e, but is 1-1, onto on H_e.

$V_e = \ker dp.$

ie) dp annihilates V_e, but restricts to an iso on H_e.

$$dp : H_e \xrightarrow{\sim} T_b B$$

<u>dp is 1-1 on H_e</u>: Since if $v \in H_e$ had $dp(v) = 0$, then it would have to be vertical, so $v \in H_e \cap V_e = \{0\} \Rightarrow v = 0$.

<u>dp maps H_e onto $T_b B$</u>: Pick a local trivialization:

$$\begin{array}{ccc} P|_U & \xrightarrow[\sim]{t} & U \times G \\ p \downarrow & & \downarrow p' \\ b \in U & \xrightarrow{1} & U \end{array}$$

p' is onto, and $p \cong p'$.

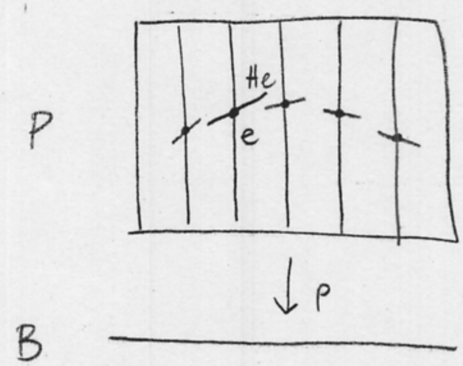
$p' = \text{id}$ on 1st entry, throw out other entry dp maps $T_e P$ onto $T_b U$ since dp' maps $T_e(U \times G)$ onto $T_b(U)$.

$$dp'(v,w) = v, \quad v \in T_b(U)$$

and bundles
$$\begin{array}{ccc} P|_U & \text{is iso to} & U \times G \\ \downarrow & & \downarrow \\ U & & U \end{array}$$

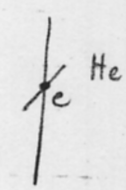
② we need H_e to vary smoothly with e (depends smoothly on $e \in P$).

③

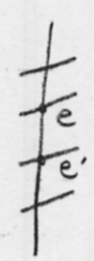


$G = \mathbb{R}$

It would be nice if once we said is horizontal, then on the same fiber,



we get all things horiz. to look the same.



ie) If e, e' are in the same fiber, we want H_e and $H_{e'}$ to be "parallel"

- horiz g, vert. part.

something horiz, something ...

<u>Recall</u>: G has a right action on P, preserving each fiber.

Moreover, given any e, e' in the same fiber, $\exists ! \; g \in G$ st

$$eg = e' \qquad \text{(right action)}$$

This is because each fiber is "like" (iso, but not in a god-given way) G w/ action like right mult.

Then – g acts on P, giving a map f

$$f : P \longrightarrow P \quad \text{and thus} \quad f(e) = e'$$

$$df : T_e P \longrightarrow T_{e'} P$$

and we require that

$$df : H_e \longrightarrow H_{e'} .$$

<u>Definition</u>: A <u>connection</u> on our principal G-bundle $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ is a choice of "horizontal" subspace $H_e \subseteq T_e P \;\; \forall e \in P$ st

① $H_e \oplus V_e = T_e P$

② H_e varies smoothly w/ e

③ If $e' = f(e)$ where $f : P \to P$ comes from right action of $g \in G$ on P, then $df(H_e) = H_{e'}$.