All Gauge Theory stuff can be found in <u>Gauge Fields, Knots &, Gravity</u>.

4/8/02

Goal: Categorify Gauge Theory

<u>Gauge Theory</u> involves concepts of:

- groups
- Lie groups
- Lie algebras
- bundles
- connections on bundles
- curvature of a connection
- Yang-Mills eqns – describe forces of nature <u>other</u> than gravity

all sets w/ various properties

categorification

An example of gauge theory is gravity.

- 2-groups
- Lie 2-groups
- Lie 2-algebra
- 2-Bundles
- 2-Connections on 2-bundles
- 2-curvature on a 2-conn.
- 2-Yang Mills eqns

all categories

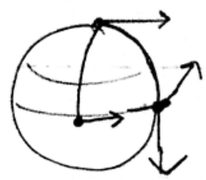
In gauge theory – think of particles as points and move particles.

How does the particle change as we move it.

· parallel transport – move an object w/out rotating it

A connection has <u>curvature</u> if you move a particle from pt A to B 2 different ways & end up w/ different things. (orientation of end result is path dependent)

Ex)



move tangent down side of sphere to get ↕

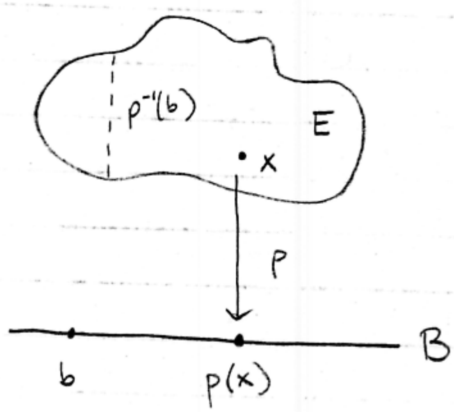
move down, then along equator & get ⟶

The group for all forces other than gravity:

$$SU(3) \times SU(2) \times U(1)$$

Bundles

Defn: A bundle is a smooth map $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ where E, B are smooth manifolds.



We call E the total space and call B the base space p the projection.

We call $p^{-1}(b)$ the fiber over b. So E is the collection of all these fibers — a bundle of fibers.

Define: A morphism of bundles from $p: E \longrightarrow B$ to $p': E' \longrightarrow B'$ to be:

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

maps $\bar{f}: E \longrightarrow E'$, $f: B \longrightarrow B'$ st this diagram commutes.

An <u>isomorphism</u> of bundles is a morphism with an inverse:

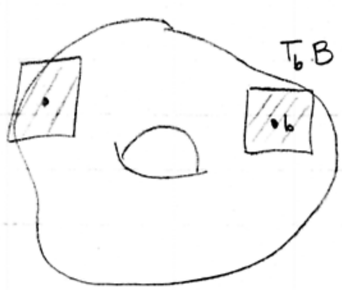
$$\begin{tikzcd} E' \arrow[r, "\bar{f}^{-1}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f^{-1}"] & B \end{tikzcd}$$

this diagram commutes when the first one above does.

We can also fix B and consider bundles over it & morphisms like this:

$$\begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "1_B = id_B"] & B \end{tikzcd}$$

<u>Tangent</u> <u>Bundles</u>:



At each pt – have tangent v. space

tangent bundle = union of tangent spaces.

- B is a manifold
- $E = \bigcup_{b \in B} T_b B$ — tangent space to B at b.

E is a manifold called TB, the <u>tangent bundle</u>.

$p : E \longrightarrow B$ sends all of $T_b B$ to b.

Example: (A trivial bundle)

Given any manifold B and any manifold F (the "fiber") we can build a bundle

$$\begin{array}{ccc} E = B \times F & \ni & (b, f) \\ \downarrow p & & \downarrow \\ B & & b \end{array}$$

We call E (the product of 2 spaces) a trivial bundle.

Note — many bundles are isomorphic to trivial bundles and not necessarily a trivial bundle itself. We call such bundles trivializable.

Defn: We call a bundle trivializable if it is isomorphic to a trivial bundle.

So, $p : E \to B$ is trivializable if we can find F and

$$\begin{array}{ccc} E & \xrightarrow{\bar{f}} & B' \times F \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{f} & B' \end{array}$$

st $f, \bar{f}$ have inverses.

f is smooth, 1-1, onto, smooth inverse — so we can make $B' = B$.

$$\begin{tikzcd}
E \arrow[r, "\bar{f}"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B' \times F \arrow[r, "f^{-1} \times id_F"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \times F \arrow[d, "\pi"] \\
B \arrow[r, "f"] & B' \arrow[r, "f^{-1}"] & B
\end{tikzcd}$$

each square is an isomorphism of bundles, so we can compose the boxes to get:

$$\begin{tikzcd}
E \arrow[r, "\bar{f} \circ (f^{-1} \times id_F)"] \arrow[d, "p"'] & B \times F \arrow[d, "\pi"] \\
B \arrow[r, "id"] & B
\end{tikzcd}$$

(opposite of usual composition)

<u>Moral</u>: a bundle is trivializable iff we can find an isomorphism:

any fiber over b gets mapped to the corresponding fiber

$$\begin{tikzcd}
E \arrow[r, "g"] \arrow[d, "p"'] & B \times F \arrow[d, "\pi"] \\
B \arrow[r, "1_B"] & B
\end{tikzcd}$$

We'll call g a <u>trivialization</u> of $p : E \longrightarrow B$.

There are many g's we can have to make a bundle trivializable.

<u>Example</u>: Consider $B = T^2$, $E = TB$

torus

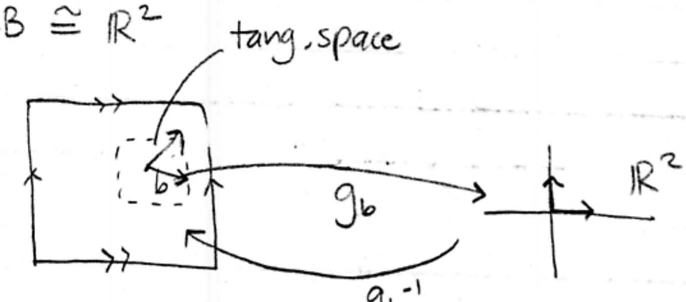
A trivialization of this bundle would be:

$$\begin{tikzcd} TB \arrow[r, "g"] \arrow[d, "p"'] & B \times \mathbb{R}^2 \arrow[d] \\ B \arrow[r, "1_B"] & B \end{tikzcd}$$

needs to be a 2-dim'l v. space

since $T_b B \cong \mathbb{R}^2$

$T^2 =$ (torus)



A trivialization $g : TB \longrightarrow B \times \mathbb{R}^2$ must send

$$T_b B \text{ to } \{b\} \times \mathbb{R}^2 \cong \mathbb{R}^2$$

so we can get

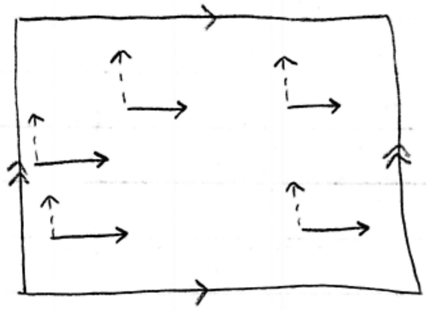
$$g_b : T_b B \xrightarrow{\cong} \mathbb{R}^2 \quad \text{(iso)}$$

Varying smoothly w/ b. Conversely any such thing gives a trivialization.

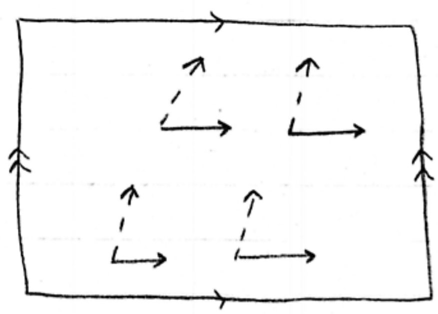
g_b^{-1} sends basis of $\mathbb{R}^2$ to T^2. We'll know what g_b^{-1} does when we know what it does on the basis.

If g_b is linear ∧ We can draw g_b^{-1} by drawing an ordered basis of $T_b B$ varying smoothly with B.

Let's draw a bunch of trivializations where g_b is linear $\forall\ b \in B$.



(A trivial bundle — "the" fiber — all fibers over all pts are the same. In our ex) $= \mathbb{R}^2$)



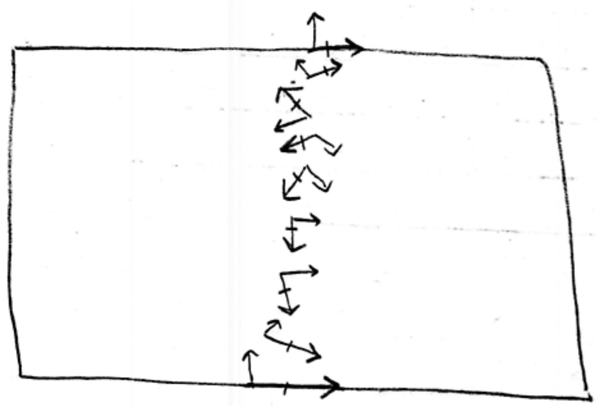
recall — pts on left = right boundary top = bottom boundary.

We can continuously (smoothly) deform top picture to bottom — ie) we have a homotopy of trivializations

These 2 trivializations are homotopic: we say 2 trivializations g_0 & g_1 are homotopic if we can find trivializations g_t $0 \le t \le 1$

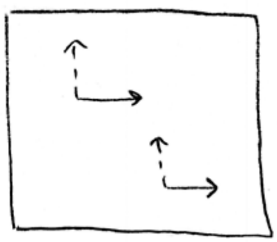
st $g_y : TB \times [0,1] \longrightarrow B \times \mathbb{R}^2$ is smooth. ($y \in [0,1]$)

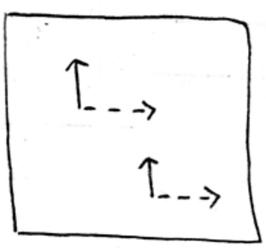
Example— this is not hamotopic to our above example.



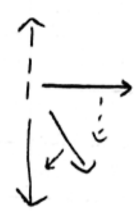
This trivialization has a 2π twist as we go up the page. We could also do 2π n rotations. All of these would be non hamotopic.

Example: these are not hamotopic





Trying



move bottom arrow around, no longer a basis!

* Left-handed basis isn't hamotopic to right-handed one.

Note- given any 2 linear trivializations g, g' of TB (B the torus) they are related by means of a smooth map $f: B \to GL(2)$

$GL(n) =$ $n \times n$ invertible matrices

$$\begin{tikzcd}
B \times \mathbb{R}^2 \arrow[d] & TB \arrow[l, "g'"'] \arrow[r, "g"] \arrow[d] & B \times \mathbb{R}^2 \arrow[d] \\
B & B \arrow[l, "1"'] \arrow[r, "1"] & B
\end{tikzcd}$$

$$g' \circ g^{-1} : B \times \mathbb{R}^2 \longrightarrow B \times \mathbb{R}^2$$
$$(b, v) \longmapsto (b, h(b,v))$$

Know 1st entry has to be b, since if we map it down to B, then right, via identities, we get b.

$h(b,v)$ is linear in v, so

$$h(b,v) = \underbrace{f(b)}_{GL(2)} v \quad \text{which gives } f: B \to GL(2).$$

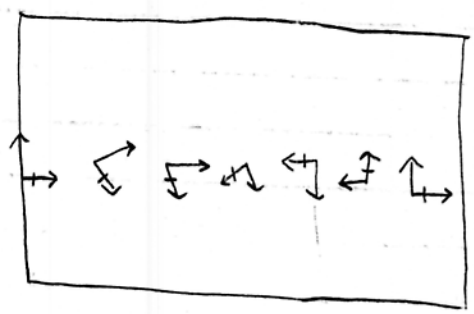
The converse of the note is also true.

Thinking a bit more, we get:

Thm: Classifying homotopy classes of linear trivializations of TB ($B = T^2$) is the same as classifying homotopy classes of maps $f: B \longrightarrow GL(2)$.

$GL(2)$ consists of terms w/ $\det = 1, -1$
these are the 2 types – twists, reflections

We could have the twists moving from left to right:



Is this another kind of 2π twist? Is this homotopic to the other one?

Check

$$\begin{array}{ccc} \pi_1(B) & \longrightarrow & \pi_1(GL(2)) \approx \pi_1(O(2)) \\ \wr\wr & & \approx \mathbb{Z} \\ \mathbb{Z} \times \mathbb{Z} & & \end{array}$$

so for
$$\begin{array}{ccc} \mathbb{Z} + \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ (x,y) & \longmapsto & ax+by \end{array}$$

where x, y viewed as the measurements of "twistiness" in x direction & y is the twistiness in y direction.