5/14/02

We saw we can define a <u>group object</u> in any <u>Cartesian category</u>, i.e. any category w/ finite product (equiv. to a category w/ binary products and a terminal object).

We can also define a <u>homomorphism</u> $f: G \to G'$ between group objects:

· should preserve mult., inverses, identities

①
$$\begin{tikzcd} G \times G \arrow[r, "m"] \arrow[d, "f \times f"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "f"] \\ G' \times G' \arrow[r, "m'"] & G' \end{tikzcd}$$

preserves mult.

②
$$\begin{tikzcd} \underline{1} \arrow[r, "i"] \arrow[d, "1"'] \arrow[dr, phantom, "\circlearrowright"] & G \arrow[d, "f"] \\ \underline{1} \arrow[r, "i'"] & G' \end{tikzcd}$$

preserve identity

③
$$\begin{tikzcd} G \arrow[r, "inv"] \arrow[d, "f"'] & G \arrow[d, "f"] \\ G' \arrow[r, "inv'"] & G' \end{tikzcd}$$

preserves inverses

Secretly - these commutative diagrams involve a natural transformation.
Even more secretly - the group objects are functors.

Given a Cartesian category, we get a new category:

<u>Defn:</u> If C is a Cartesian category, let <u>C-Grp</u> be the category whose objects are grp. objects in C and whose morphisms are homomorphisms.

A grp obj. in the cat. of sets is a group.

	C	C-Grp	
	Set	Grp	(groups)
	Top	Top Grp	(top groups mult is cont.)
	Diff	Diff-Grp	(Lie Groups)
(1)	Grp	Ab-Grp (Grp-Grp)	(abelian grps)
(2)	Ab-Grp	(Grp-Grp-Grp) (Ab-Grp-Grp)	(abelian grps)
		Ab-Grp	(same as above)
(3)	Vect products = $\oplus$ term. obj = 0	Vect	
	Top-Grp	Ab-Top-Grp	(abelian top. grps)
	Diff-Grp	Ab-Diff-Grp	(abelian Lie grp)
(4)	Cat	Cat-Grp	(categorical grps) (*)

(1) What's a group object in Grp?

A group G w/ homomorphism: $m: G \times G \longrightarrow G$
(which is <u>not</u> the mult. in the group)
(the original mult isn't a homo.)

unit map: $i: \underline{1} \longrightarrow G$ "unit homomorphism"

the 1-elt group (terminal grp)

(think of it as picking out a particular elt of G)

Since i is a homo, must send identity elt of group
maps
$\underline{1} \in \underline{1}$ to $1 \in G$. (no new data)

and "inv" homo: $\text{inv}: G \longrightarrow G$

satisfying axioms of a group object.

m is a homo:

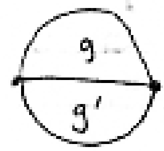
$$m(gg', hh') = m((g,g')(h,h')) = m(g,g')\,m(h,h'), \quad g,g',h,h' \in G$$

Let $m(g,g') = g \cdot g'$.

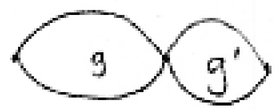
So we get:

$$(gg')\cdot(hh') = (g\cdot g')(h\cdot h')$$

In a 2-category we have 2 forms of "composition."



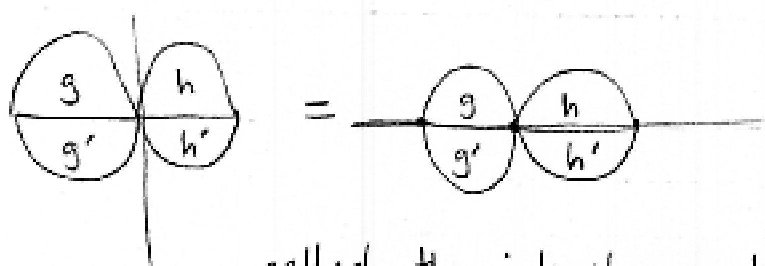
gg'



$g \cdot g'$

so the law on the prev. pg looks like:

$$(gg') \cdot (hh') = (g \cdot g')(h \cdot h')$$

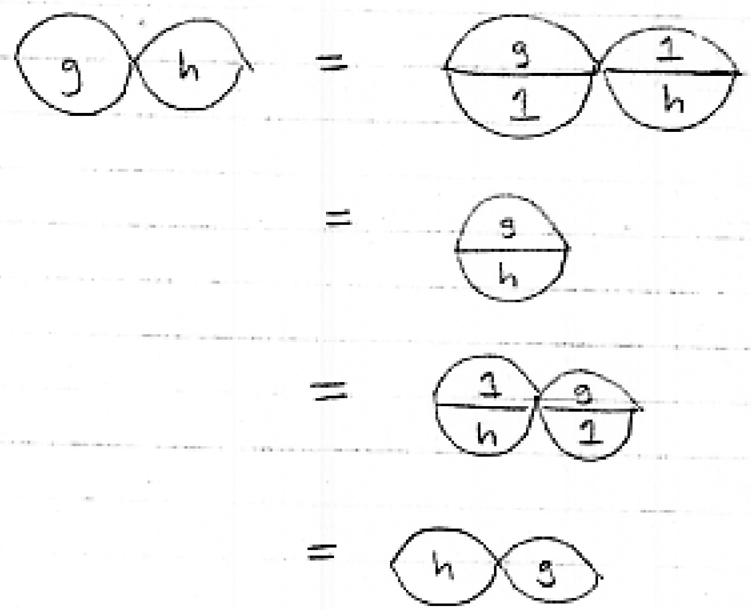


called the <u>interchange</u> <u>law</u>.

Guess: A group object in <u>Grp</u> is a 2-cat w/ only one object, and only one morphism – i.e. some kind of commutative monoid.

These 2 multiplications are the same!

The identity $\underline{1}$ is the identity for the old & new (horiz & vert) mult:



$\Rightarrow gh = g \cdot h = hg = h \cdot g$

SO – there is only one mult & it is commut., so we get a commut. grp.

(2) argument same as above; doesn't change if G above was abelian

(3) A v. space V w/ linear map $V \times V \longrightarrow V$, etc...

same argument: new mult = add. in v. space

but we have no extra structure – nothing new, and new mult = addition in v. space.

* A group object in <u>Cat</u> is a category object in <u>Grp</u>.

(4) grp objects in the category of categories

(*) <u>categorical</u> <u>groups</u> = strict <u>2-groups</u>

= (strict 2-categories w/ one object and all morphisms & 2-morphisms invertible)
(this defn is equiv to categorical grps)

A grp is a special kind of category, so let's talk about:

<u>Category</u> <u>Objects</u>

What's a category object in some category C?

It will have:

- an object $O \in C$ of <u>objects</u>
- an object $M \in C$ of <u>morphisms</u>
- <u>source</u> & <u>target</u> <u>morphisms</u> $s, t : M \longrightarrow O$

$$sf \xrightarrow{f} tf$$

- <u>identity</u> $i : O \longrightarrow M$ (identity-morphism) morphism
$x \longmapsto 1_x$

· <u>composition morphism</u> : we can only compose composable morphisms

(pullback – subset of a product)
(pushout – quotient)

$$\circ : M\ {}_{t}\!\times_{s} M \longrightarrow M$$ defined only on the <u>pullback</u>

$$\begin{tikzcd}
M\ {}_{t}\!\times_{s} M \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[d, "s"] \\
M \arrow[r, "t"'] & O
\end{tikzcd}$$

means this is a pullback square.

such that the following commute: (defn of category)

①
$$\begin{tikzcd}
O \arrow[r, "i"] \arrow[dr, "1"'] & M \arrow[d, shift right, "s"'] \arrow[d, shift left, "t"] \\
& O
\end{tikzcd}$$

② source / target of composition must work out

$$\begin{tikzcd}
M\ {}_{t}\!\times_{s} M \arrow[r, "\circ"] \arrow[d, "p_1"'] & M \arrow[d, "s"] \\
M \arrow[r, "s"'] & O
\end{tikzcd}
\qquad
\begin{tikzcd}
M\ {}_{t}\!\times_{s} M \arrow[r, "\circ"] \arrow[d, "p_2"'] & M \arrow[d, "t"] \\
M \arrow[r, "t"'] & O
\end{tikzcd}$$

③
$$\begin{tikzcd} O \,{}_1\!\times_s M \arrow[r, "i \times 1"] \arrow[dr, "p_2"'] & M \,{}_t\!\times_s M \arrow[d, "\circ"] \\ & M \end{tikzcd}$$

- composing on right by identity
  $f \circ 1 = f$

$$\begin{tikzcd} M \,{}_t\!\times_1 O \arrow[r, "1 \times i"] \arrow[dr, "p_1"'] & M \,{}_t\!\times_s M \arrow[d, "\circ"] \\ & M \end{tikzcd}$$

- composing on left by identity $1 \circ f = f$

④ associativity

$$\begin{tikzcd} M \,{}_t\!\times_s M \,{}_t\!\times_s M \arrow[r, "1 \times \circ"] \arrow[d, "\circ \times 1"'] & M \,{}_t\!\times_s M \arrow[d, "\circ"] \\ M \,{}_t\!\times_s M \arrow[r, "\circ"'] & M \end{tikzcd}$$

* Moral: we can define "category object in C" if C has pullbacks.

<u>Prop</u>: A category has finite limits iff it has pullbacks and terminal objects.
(These give finite products.)

We can also define <u>functors</u> (functor-morphism) between category objects:

$F: (O,M) \longrightarrow (O',M')$ consists of morphism

$$F_o : O \longrightarrow O'$$

$$F_m : M \longrightarrow M'$$

st the usual properties of a functor hold.

<u>Defn</u>: Given a category w/ pullbacks C, we let C-<u>Cat</u> be the category where objects are category objects in C and morphisms are functors between those.

Note: often people use "topological category" when O is discrete.

	$\underline{C}$	$\underline{C\text{-Cat}}$
	Set	Cat
(1)	Top	Top-Cat $\ni$ "topological category" (top. space of objects, space of morphisms, comp. is cont.)
(2)	Diff	Diff-Cat $\ni$ "smooth category" (Ehresman: "differentiable category")

(1) an example of a topological category is the <u>fundamental groupoid</u>.

X Here $O = X$

or – $\Pi_1^{\text{thin}}(M)$

(2) But Diff doesn't always have pullbacks.

$$\begin{tikzcd} ? & A \arrow[d] \\ B \arrow[r] & X \end{tikzcd}$$

But we just need the pullbacks to exist that we're going to use.

Ex)

$$\begin{tikzcd} & ? \arrow[dl] \arrow[dr] & \\ \mathbb{R}^2 \arrow[dr] & & \underline{1} \arrow[dl] \\ & \mathbb{R} & \end{tikzcd}$$

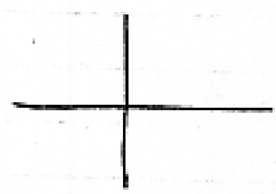
($\{pt\}$ - 0 dim'l manifold)

$(x,y) \mapsto xy$

Hope $? \subseteq \mathbb{R}^2 \times \underline{1}$

$? = \{ (x,y) \mid xy = 0 \} \subseteq \mathbb{R}^2 \times \underline{1}$

$\{ xy = 0 \} \in \mathbb{R}^2$



not a smooth submanifold