5/16/02

C-Cat = the category whose objects are category objects in C and whose morphisms are functors (functor-morphisms) between these.

C	C-Cat
Set	Cat
① Top	Top-Cat
② Diff	Diff-Cat
③ Grp	Grp-Cat
④ Ab Grp	Ab Grp-Cat (2-term chain complex of abel. groups)
⑤ Vect	Vect-Cat (2-term chain complexes of v. spaces $X_0 \xleftarrow{d} X_1$)
⑥ Lie Grp = grp object in Diff = Diff Grp	[Diff-Grp]-Cat = (strict) Lie 2-grps
⑦ Cat	Cat-Cat = double category

① space of objects, space of morphisms
"topological categories"
(people usually assume the space of objects is discrete)

② problem! Diff doesn't always have pullbacks

$X \in C\text{-Cat}$ has an object of objects, $O = X_0 \in C$
object of morphisms, $M = X_1 \in C$

composition: need source & target to match
(target of 1st = source of 2nd)

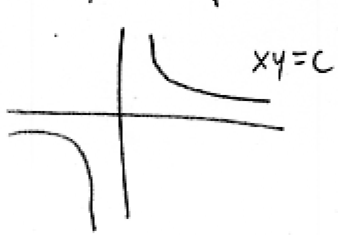
$\circ \quad X_1 \times_{t\, X_0\, s} X_1 \longrightarrow X_1$

Ex)

$$\begin{tikzcd}
& \{(x,y) \mid xy = c\} \arrow[dl] \arrow[dr] & \\
\mathbb{R}^2 \arrow[dr, "(x,y) \mapsto xy"'] & & \{*\} \text{ (1-pt manifold)} \arrow[dl, "* \mapsto c"] \\
& \mathbb{R} &
\end{tikzcd}$$

Want pairs of something in $\mathbb{R}^2$, something in $\{*\}$

If $xy = c$ parabola



but at (0,0), deriv of map $(x,y) \longmapsto xy$ is zero.

- Pullback exists in Diff only if $c \neq 0$.
- transversality — condition that guarantees pullbacks.

* In the defn. of category object — we demand that the necessary pullbacks exist.

③ called a "groupal category" = 2-group (strict)

recall – we talked about a categorical group (a group object in Cat)

④

$$X_0 \xleftarrow{d} X_1 \qquad X_1, X_0 \text{ abelian grps}$$

(don't need $d^2 = 0$ since only 1 map)

⑤ categorify Gauge Theory — categorified Vector spaces.

We've mentioned 2 concepts:

- "group objects in Cat" = "categorical groups"
- "category objects in Grp" = "groupal categories"

These are the same!

<u>Thm</u>: Grp-Cat $\simeq$ Cat-Grp

equivalence of categories

(*) follows from commutativity of exponentiation.

proof: This is an example of "commutativity of abstraction." Both "grp object" and "category object" exist in any category w/ finite limits.

More generally — suppose X and Y are structures that we can define using commut. diagrams w/ finite limits. Then C-X (an X object in C) and C-Y (a Y-object in C) both exist and have finite limits when C does.

Then we can talk about a Y object in $(C\text{-}X)$ and X object in $(C\text{-}Y)$. So

$$(C\text{-}X)\text{-}Y \simeq (C\text{-}Y)\text{-}X \qquad (*)$$

Here: C-X is the category of X-objects in C, etc.

In fact, there is a category w/ finite limits $\underline{X}$, called "the walking X" or "the Platonic idea of X."

Suppose X is the concept of <u>group</u>. Then the walking group, ie) $\underline{X}$ is a category w/ finite limits containing:

1) an object G (idea of a grp)

2) a morphism $m: G \times G \longrightarrow G$

3) a morphism $i: 1 \longrightarrow G$ (1 is terminal obj)

4) a morphism $inv: G \longrightarrow G$

satisfying exactly the relations in defn of group:

1) l/r unit laws
2) associative law
3) l/r inverse laws

written out as commut diagrams

and their consequences.

(This is similar to defining something via generators & relations.)

For more precision — see the concepts of <u>sketch</u> and <u>finite limits theory</u> in Barr & Wells Toposes Triples & Theory

$\underline{X}$ is also called the "theory of groups." (theory of <u>a</u> group)

* A group object in a category C w/ finite limits turns out to be precisely a functor

$$F : \underline{X} \longrightarrow C$$

which preserves finite limits, or is <u>left</u> <u>exact</u>

A homomorphism between 2 group objects (in C): $F, F' : \underline{X} \to C$ is any natural transformation,

$$\alpha : F \Rightarrow F'$$

So C-X (X objects in C) is the category called:

$$C\text{-}X = \mathrm{Lex}(\underline{X}, C) \qquad \text{(left exact)}$$

where objects in $\mathrm{Lex}(\underline{X}, C)$ are left exact functors $F : \underline{X} \to C$ and morphisms are nat. transf between them.

(Note – Lex is a hom cat)

We wanted to show:

$$(C\text{-}X)\text{-}Y \simeq (C\text{-}Y)\text{-}X$$

so we need:

$$\mathrm{Lex}(\underline{Y}, \mathrm{Lex}(\underline{X}, C)) \simeq \mathrm{Lex}(\underline{X}, \mathrm{Lex}(\underline{Y}, C))$$

(seen before when Lex was hom, $\underline{X}, \underline{Y}$, C were R-modules)

Given categories C & D w/ finite limits, there is a category w/ finite limits called $C \otimes D$ st

$$\mathrm{Lex}(C \otimes D, E) \simeq \mathrm{Lex}(C, \mathrm{Lex}(D, E))$$

E another cat w/ finite limits

But $C \otimes D \simeq D \otimes C$, so —

(Here — $C \otimes D$ starts w/ $C \times D$ &, throw in all finite limits.)

Since $C \otimes D \simeq D \otimes C$, we get

$$\mathrm{Lex}(\underline{Y}, \mathrm{Lex}(\underline{X}, C)) \simeq \mathrm{Lex}(\underline{Y} \otimes \underline{X}, C)$$
$$\simeq \mathrm{Lex}(\underline{X} \otimes \underline{Y}, C)$$
$$\simeq \mathrm{Lex}(\underline{X}, \mathrm{Lex}(\underline{Y}, C)) \quad \square$$

In fact:

<u>Thm</u>: TFAE

① Category Objects in Grp

② Group objects in Cat

③ strict 2-grps (from last quarter)
ie— strict 2-cats w/ one object and all morphisms and 2-morphisms invertible

④ crossed modules.

(try using defns)

We've seen ① & ② are equivalent.

What about ① & ③?

Let C be a category object in Grp. It has a group of objects C_0. But we can compose grp objects, so we'll draw them as arrows.

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd} \qquad g \in C_0$$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \arrow[r, bend left, "h"] & \bullet \end{tikzcd} \qquad \text{gives } gh \in C_0.$$

It also has a group of morphisms, C_1 and source & target homomorphisms:

$$s, t : C_1 \longrightarrow C_0$$

We draw $h : g \longrightarrow g'$ $(h \in C_1)$, $g, g' \in C_0$ as

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g = s(h)"] \arrow[r, bend right=60, "g' = t(h)"'] \arrow[r, phantom, "\Downarrow h"] & \bullet \end{tikzcd}$$

Given $h, h' \in C_1$, we draw their product in C_1 as

* this is mult. in the group C_1

$$\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow h"] & \bullet \arrow[r, bend left=60] \arrow[r, bend right=60] \arrow[r, phantom, "\Downarrow h'"] & \bullet \end{tikzcd}$$

since s, t should be homos.

so: $$s(hh') = s(h)\, s(h')$$
$$t(hh') = t(h)\, t(h')$$

* this is <u>composition</u> of morphisms in C_1.

Given $h: g \to g'$, $h': g' \to g''$ we <u>compose</u> these morphisms and draw them vertically

$$\begin{tikzcd} \bullet \arrow[rr, bend left=60, "g"] \arrow[rr, "g'" description] \arrow[rr, bend right=60, "g''"'] \arrow[rr, phantom, shift left=3ex, "\Downarrow h"] \arrow[rr, phantom, shift right=3ex, "\Downarrow h'"] & & \bullet \end{tikzcd}$$

(here need target of 1st to be source of 2nd)

How does the interchange law follow?

* It follows from the fact that composition is a homomorphism!

of morphisms in our cat. object is a homo

$$(ff')\cdot(gg') = (f\cdot g)(f'\cdot g')$$