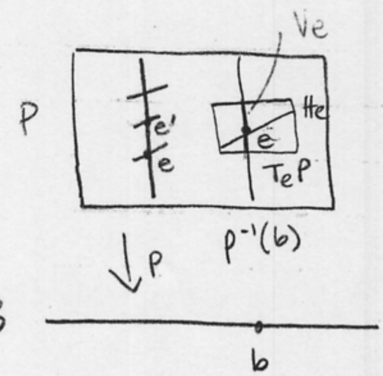
5/2/02 **Connections on Principal Bundles**
(as opposed to vector bundles)

G acts as transformations of P. Each fiber looks like G.

Let B be a manifold – "base space"
Let G be a Lie group – "gauge grp" (physics), "structure grp" (math)

Let $\begin{array}{c} P \\ \downarrow p \\ B \end{array}$ be a principal G-bundle.



V_e gets mapped to zero by projection.

$$V_e = \ker dp = \{ v \in T_e B \mid dp(v) = 0 \}$$

A connection gives us a way to pick H_e (horiz. space) orthogonal to V_e (vertical space).

Defn: A connection H on $\begin{array}{c} P \\ \downarrow \\ B \end{array}$ is a family of subspaces

$H_e \subseteq T_e P$ st

① $T_e P = V_e \oplus H_e$

② H_e varies smoothly w/ e.

③ Want $H_e = H_{e'}$ if e, e' are in same fiber. ie –

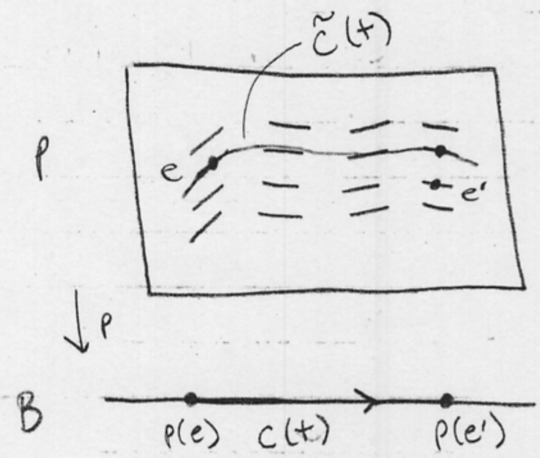
③ $H_{f(e)} = df(H_e)$ where $f: P \to P$ is given by right action of G on P.

$f(e) = eg$, $g \in G$

so $df: T_eP \to T_{f(e)}P$.

* w/out a connection, there is no way to relate a point in one fiber to a point in another fiber.

But w/ a connection, we can "carry" a pt from one fiber to another along a path. (called <u>parallel transport</u>)



carry points "horizontally"

pick a path in B, get a path in P using the connection.

* process of carrying involves making a choice

get path in P by lifting path in B (ie— path in P gets mapped down to path in B)

Parallel Transport:

* depends on path chosen in B and connection chosen in P.

To compare 2 points $e, e' \in P$ we need to pick a path from $p(e)$ to $p(e')$ in B and pick a path in P whose tangent vector is always horizontal which "lifts" this path (ie - projects down to path in B) and starts at e, then ask - does it end at e'?

Defn: Given a path $C : [a,b] \longrightarrow B$ and a bundle $\begin{array}{c} E \\ \downarrow \\ B \end{array}$, we say the path

$\tilde{C} : [a,b] \longrightarrow E$ lifts C (or is a lift of C)

if this diagram commutes:

ie. $p(\tilde{c}) = C$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ {[a,b]} \arrow[ur, "\tilde{c}"] \arrow[r, "c"'] & B \end{tikzcd}$$

<u>Defn</u>: Given a principal bundle $\begin{array}{c} P \\ \downarrow \\ B \end{array}$ w/ connection H on it,
we say a path
$\tilde{c} : [a,b] \longrightarrow P$
is <u>horizontal</u> if
$\forall t \in [a,b]$

$$\tilde{c}'(t) \in H_{\tilde{c}(t)}$$

tangent vector

<u>Thm</u>: Suppose $\begin{array}{c} P \\ \downarrow \\ B \end{array}$ is a principal bundle w/ connection H on it and suppose

$c : [a,b] \longrightarrow B$. Then there exists a unique path
$\tilde{c} : [a,b] \longrightarrow P$ such that

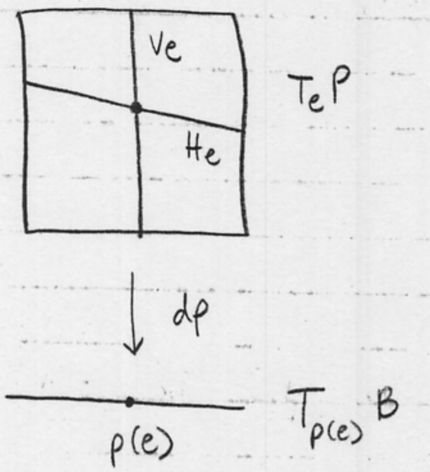
① $\tilde{c}$ lifts c
② $\tilde{c}$ is horizontal
③ $\tilde{c}(a)$ is a given point $e \in p^{-1}(c(a))$

* when lifting, there are many places to start $\tilde{c}(t)$, but not if we specify the starting pt of $\tilde{c}(t)$.

V_e - god-given (ker of dp)

H_e - we pick

<u>Sketch of proof</u>: We claim that $\tilde{c}'(t)$ is uniquely determined from ① - ③.



$$\begin{tikzcd}
T_eP \arrow[d, "dp"'] & = & V_e \oplus H_e \arrow[d] \arrow[d, shift left=6, "\wr"] \\
T_{p(e)}B & = & 0 \oplus T_{p(e)}B
\end{tikzcd}$$

(last time we showed the map from $H_e \longrightarrow T_{p(e)}B$ is 1-1, onto)

So any $u \in T_eP$ is the same as a pair $u = (v, h) \in V_e \oplus H_e$ and h is determined by $dp(h) \in T_{p(e)}B$.

So, to give me u, you need to give v and $dp(u) = dp(h)$

To know $\tilde{c}'(t)$, we just need to know its vertical part and $dp(\tilde{c}'(t))$.

Since $\tilde{c}$ is horizontal, its vertical part is 0. We know by ① (that $\tilde{c}$ lifts c) so we know

$$dp(\tilde{c}'(t)) = (p \circ \tilde{c})'(t) \underset{\text{by ①}}{=} c'(t)$$

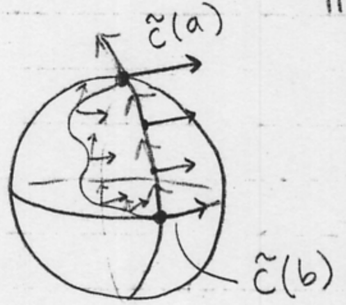
$p \circ (\tilde{c}'(t)) = c'(t)$

From ③, since we have an initial condition, and know $\tilde{c}'(t)$ the Fund Thm of ODE's says we can find $\tilde{c}(t)$ uniquely. (Note— we haven't said $\tilde{c}(t)$ exists.)

For existence we need more work, but it's true.

Example: $B = S^2$, $P = FS^2$ (frame bundle).
Usual "round" metric on S^2 gives P a connection
Here's what it looks like to find a horizontal lift $\tilde{C}$ of a path C in S^2.

other path winds up w/ different pair of arrows



We say we are "parallel transporting" our frame from $\tilde{c}(a)$ to $\tilde{c}(b)$.

(curvature)
We say this connection is curved since our answer depends on the path taken (not just endpts)

Last quarter— a connection was a functor.
(thin fund. groupoid)

curvature— measures how difficult it is to lift a surface

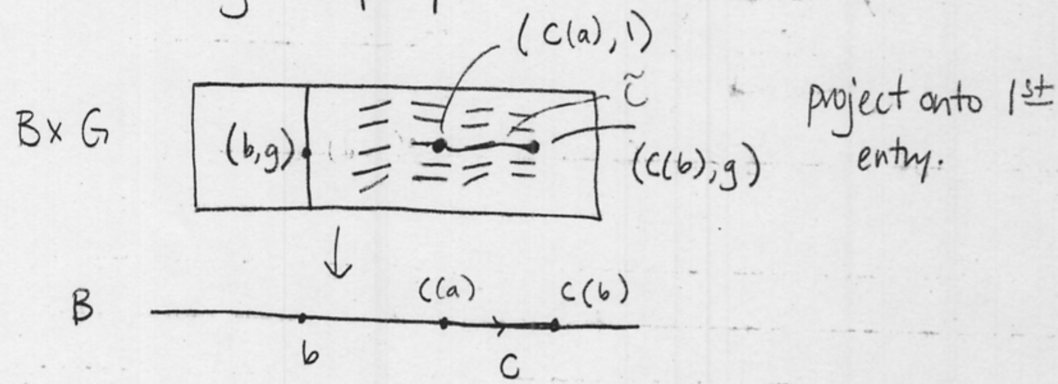
connection— tells us how to lift a curve.

We see that a connection allows us to lift any curve to a horizontal one, uniquely if we're told how to lift starting point.

Let's suppose our principal bundle $\begin{array}{c} P \\ \downarrow \\ B \end{array}$ is <u>trivial</u>

ie– $\begin{array}{c} P = B \times G \\ \downarrow \\ B \end{array}$

Then – things simplify:



Want to lift c to a horiz. path. We can lift the starting pt to the <u>identity</u> of the group G.

Given $c : [a,b] \longrightarrow B$ we can uniquely lift it to a horiz. path $\tilde{c} : [a,b] \longrightarrow B \times G$ st $\tilde{c}(a) = (c(a), 1)$ where $1 \in G$ is identity, and we get a group element g st $\tilde{c}(b) = (c(b), g)$.

Recall – Thin fund grpoid – had an equiv relation on paths

going $a \to b \to a$ is same as staying at a.

Note – g only depends on C, so we get a function

$$A : \{\text{paths in } B\} \longrightarrow G$$

sending C to $g \in G$. G is a 1-elt category (a grp) so A is a functor.

$A(C)$ doesn't depend on the parametrization of C and in fact it depends on "thin homotopy class" of C, so we get:

$$A : \pi_1^{\text{thin}}(B) \longrightarrow G$$

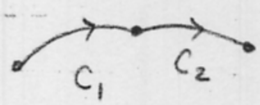
Claim: A is a functor. Show:

- $A(1_b) = 1$
- $A(C_1 C_2) = A(C_2)\, A(C_1)$

- 1_b is constant path at $b \in B$ (actually equiv. class of constant paths at b)

fact that A is a functor means:

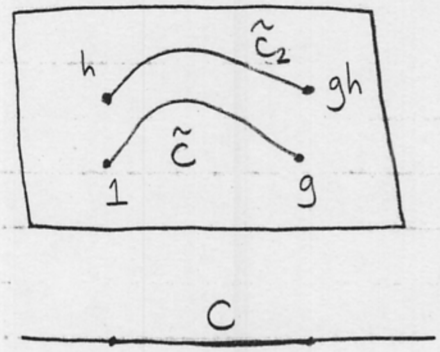
- C_1: path from b to b'
  C_2: path from b' to b''
  $C_1 C_2$: path from b to b''.



- don't move quark, stays where it is
- carry quark from here to SB
  SB to Redlands
  same as carrying quark from here to Redlands in same way.

Show $A(c_1 c_2) = A(c_2)A(c_1)$

problem: we know how to get a path from 1 to g
and then to g'.
But – not starting at 1... what do we do?



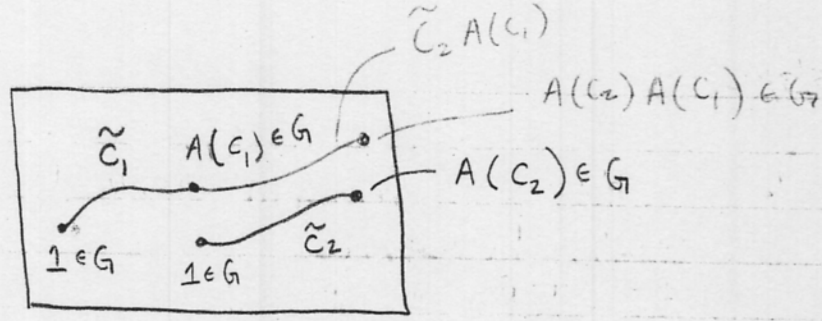
Show if lifted our path so it doesn't start at 1,
but starts at h –

ie) If we lift c to horiz $\tilde{c}_2$ w/ $\tilde{c}_2(a) = (c(a), h)$
we'd get $\tilde{c}_2(b) = (c(b), gh)$.

G acts on the right on P, preserving H, so if
our path $\tilde{c}$ is horizontal ($\tilde{c}$ starts at 1)
then $\tilde{c}_2$ will also be horizontal.

$$\tilde{c}_2(t) = \tilde{c}(t)h$$

Then – we check functorality.



we can right translate 2^{nd} piece of lifted curve.

so, we see that

$$A(C_1 C_2) = A(C_2)\,A(C_1)$$

(this is backwards from what we wanted)
so we change it...

so A is a <u>contravariant</u> functor