5/21/02

<u>TFAE</u>: (and we call them all a "strict 2-group.")

1) category object in Grp (groupal category)

2) group object in Cat (categorical grp)

3) a strict 2-category w/ one object and all morphisms & 2-morphisms invertible.

4) crossed module

Let's show that 1) is the same as 4).

Let C be a category object in <u>Grp</u>. This consists of:

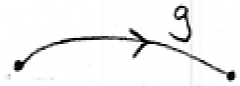
- a group of objects C_0
- a group of morphisms C_1
- source, target $s, t : C_1 \longrightarrow C_0$ group homomorphisms
- $i : C_0 \longrightarrow C_1$ grp homomorphism
- composition $\circ : C_1 \times_{C_0} C_1 \longrightarrow C_1$

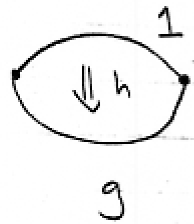
satisfying usual cat. axioms.

Now — let $G = C_0$ be the group of all objects

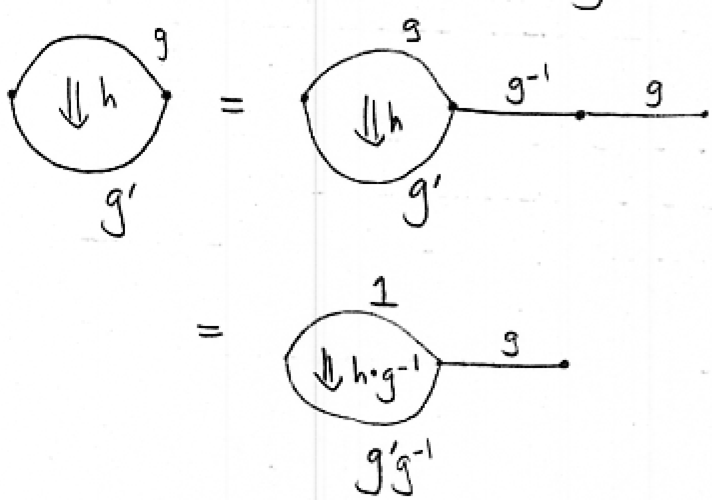
$$H = \ker s = \{h : 1 \to g\} \subseteq C_1$$

be the group of morphisms from $1 \in G$ to anything

G:  $g \in G$

H:  $h \in H$

It's okay to let $H = \ker s$ since, given something in C_1, we can express it in terms of something in H and something in G.



So, as sets, $C_1 = H \times G$.

But C_1 is a group!

Now — let $G = C_0$ be the group of all objects

$$H = \ker s = \{ h : 1 \longrightarrow g \} \subseteq C_1$$

be the group of morphisms from $1 \in G$ to anything

G:
$$\begin{tikzcd} \bullet \arrow[r, bend left, "g"] & \bullet \end{tikzcd} \qquad g \in G$$

H:
$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "1"{name=U}] \arrow[r, bend right=60, "g"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd} \qquad h \in H$$

It's okay to let $H = \ker s$ since, given something in C_1, we can express it in terms of something in H and something in G.

$$\begin{tikzcd} \bullet \arrow[r, bend left=60, "g"{name=U}] \arrow[r, bend right=60, "g'"'{name=D}] & \bullet \arrow[from=U, to=D, Rightarrow, "h"] \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left=60, "g"{name=U}] \arrow[r, bend right=60, "g'"'{name=D}] & \bullet \arrow[r, "g^{-1}"] \arrow[from=U, to=D, Rightarrow, "h"] & \bullet \arrow[r, "g"] & \bullet \end{tikzcd}$$

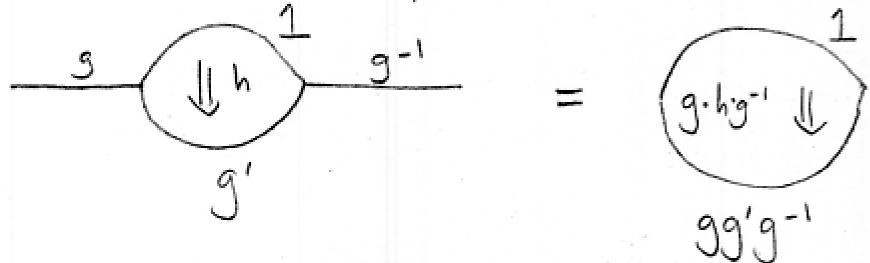
$$= \begin{tikzcd} \bullet \arrow[r, bend left=60, "1"{name=U}] \arrow[r, bend right=60, "g'g^{-1}"'{name=D}] & \bullet \arrow[r, "g"] \arrow[from=U, to=D, Rightarrow, "h \cdot g^{-1}"] & \bullet \end{tikzcd}$$

So, as sets, $C_1 = H \times G$.

But C_1 is a group!

Not as groups — In fact $C_1 = H \rtimes G$
(semidirect product)

This is the semidirect product where G acts on H by:



we have an action of G on H:

i.e. we have $\alpha : G \longrightarrow \mathrm{Aut}(H)$

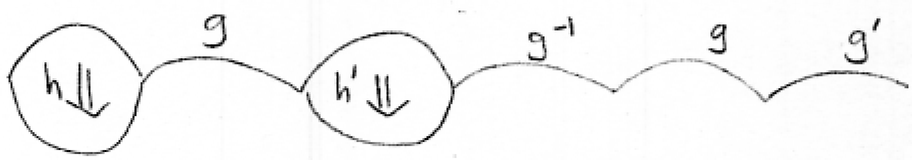
$$\alpha(g)(h) = g \cdot h \cdot g^{-1}$$

Product in C_1:

$(h,g)\,(h',g') =$

* mult. in C_1 is horizontal comp

$=$



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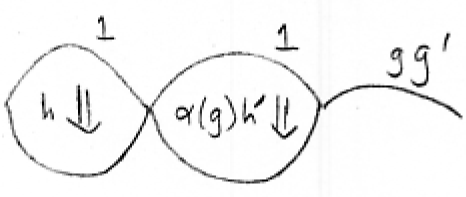


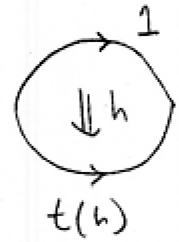
$$= (h\,\alpha(g)h' ; gg') = (h,g)\,(h',g')$$

G acting on H, create a new group $H \rtimes G$ where mult defined as:

$$(h,g)(h',g') = (h\,\alpha(g)h',\ gg')$$

where $\alpha : G \longrightarrow \mathrm{Aut}(H)$
$\alpha(g)h \longmapsto ghg^{-1}$

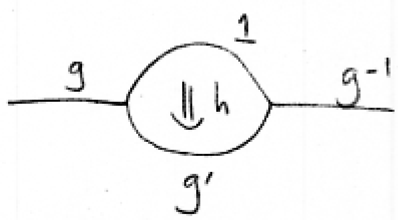
We have G, H, $\alpha : G \longrightarrow \mathrm{Aut}(H)$ and also $t : H \longrightarrow G$



<u>Defn</u>: A <u>crossed</u> <u>module</u> is a group G, a group H an action α of G on H, a homomorphism $t : H \longrightarrow G$ st

① t is G-equivariant:

$$t(\alpha(g)h) = g\,t(h)\,g^{-1}$$



$h \in H$
$g, g' \in G$

$$t(\alpha(g)h) = gg'g^{-1} = g\,t(h)\,g^{-1}$$

target

G, H are defined w/ r/t horizontal comp.
(so inverses are inverses w/r/t horiz. comp)

mult in G is , mult in H is: 

② Peiffer Identity:

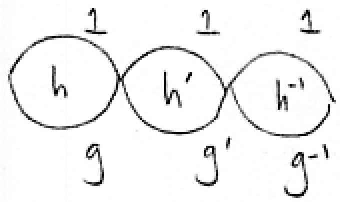
$$h h' h^{-1} = \alpha(t(h))\, h' \qquad \begin{array}{l} h \in H \\ t(h) \in G \end{array}$$

This says H acts on itself in 2 ways: conjugation & $H \xrightarrow{t} G \xrightarrow{\alpha} \mathrm{Aut}(H)$

Let's work this out as pictures.

$h : 1 \longrightarrow g, \quad h' : 1 \longrightarrow g'$

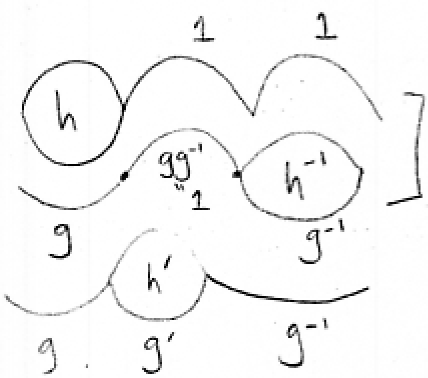
LHS:



$\overset{?}{=}$

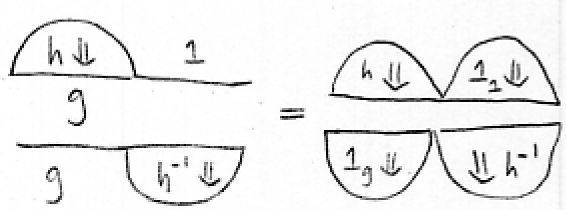
RHS

Use the interchange law

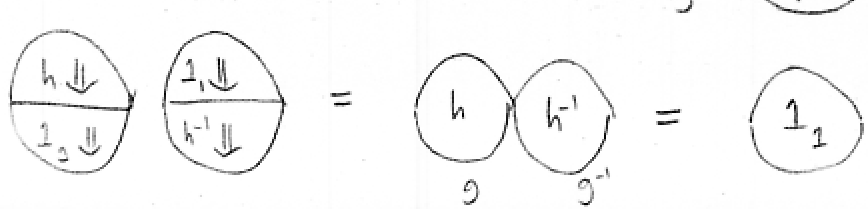
$=$ 

need to cancel, and they do!

$\|$



interchange law

$=$ 

∎

* Moral: So Peiffer identity is the interchange law.

So, given a category object in Grp, we get a crossed module. The converse is true too: you can construct a strict 2-category w/ one object and everything (morphisms, 2-morphisms) invertible from a crossed module and Peiffer identity ⇒ interchange law.

Cat objects in Grp obj in Diff

Similarly we have an equivalence between Lie 2-groups which are category objects in the category of Lie groups, or

$$(\text{Diff-Grp})\text{-Cat} \simeq (\text{Diff-Cat})\text{-Grp}$$

and Lie crossed modules.

Defn: A Lie crossed module is a crossed module where G and H are Lie groups and t, α are smooth homomorphisms.

Examples of (Lie) crossed modules:
aka (Lie) strict 2-groups:

① Let H be trivial, then α, t must be trivial, so that a crossed module is just a group G.

* a group is a special kind of 2-group

Here - we get a 2-group that's trivial on the top (morphism) level.
ie - only identity morphisms.

Moral - A grp is a 2 grp

② Suppose G is trivial. Then α, t are trivial, so that a crossed module has:

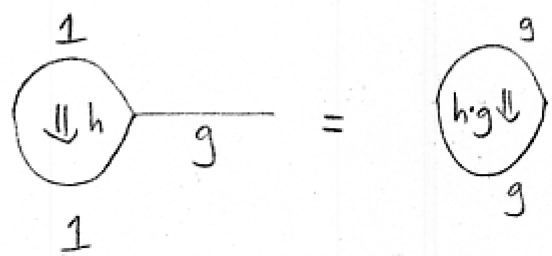
$$h h' h^{-1} = h' \quad \text{(Peiffer identity, w/ } \alpha, t \text{ trivial)}$$

so a crossed module is just an <u>abelian group</u> H (by Peiffer)! This is the Eckmann-Hilton argument: a 2-group w/ one object is an abelian group.

③ Suppose that $t: H \longrightarrow G$ is the trivial homomorphism (sends everything to one).

Then a crossed module is just a group G, an abelian group H (by Peiffer) and an action α of G on H.

$t(\alpha(g)h) = g t(h) g^{-1} \rightarrow 1 = gg^{-1}$. This tells us nothing.



This gives a <u>skeletal</u> strict 2-group.

Recall: A category is <u>skeletal</u> if isomorphic objects are equal. (ie- only isos between an object & itself)

2-groups

	weak	strict
Muscular (not necess. skeletal)	?	crossed modules
skeletal	G acting on abelian H w/associator	G acting on abelian H

$a : G^3 \to H$

identity &, inverse laws can also be weak

③ a) G a Lie Grp,
$H = V$ a V. space (an abelian Lie group)

we call an action a representation α of G on V.

So – a representation of a Lie group gives a Lie 2-group.

③ b) Given a Lie Group G, we can let it act on its Lie alg. Every Lie grp acts on its Lie alg.

$G =$ Lie grp.
$H = \mathfrak{g}$ Lie alg of G.

and α the adjoint rep of G on $\mathfrak{g}$.

the tangent 2 group of a Lie group.

$$C_1 = H \rtimes G = \mathfrak{g} \rtimes G$$

$= $ tangent bundle of G
$= TG$

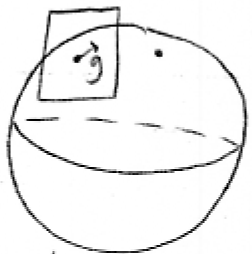
$m: G \times G \longrightarrow G$

$\downarrow$ differentiate

$dm: TG \times TG \longrightarrow TG$, so TG becomes a group

· tangent vectors – we draw as arrows – they're morphisms!

Ex)



$T(SU(2))$ $\qquad SU(2) = S^3$

· tangent vectors – really only go from g to g.

The tangent vectors are morphisms!

More generally, if M is any smooth manifold, TM becomes a skeletal smooth-category (category objects in Diff)

$v \in T_pM$ gets interpreted as $v: p \longrightarrow p$

③ c) Given a Lie group, how else can we get a representation of it?

G = Lie grp.
$H = \mathfrak{g}^*$ (dual of Lie alg) cotangent vectors

and α be the "coadjoint" rep of G on $\mathfrak{g}^*$ (dual of rep in example ③(b))

Here $\quad C_1 = T^*G$

④ Suppose α is trivial.

Then a crossed module is a group G, an abelian group H (by Peiffer) and a homo $t : H \to G$ st

$$t(h) = g\, t(h)\, g^{-1},$$ which means $t(h)$ is in the center of G. $Z(G)$

$t : H \to Z(G)$