5/23/02

Examples of strict 2-groups: (aka crossed modules)

Recall – A crossed module consists of:

1) a group G
2) a group H
3) an action α of G on H $\quad \alpha: G \longrightarrow \mathrm{Aut}(H)$
4) a homomorphism $t: H \longrightarrow G$
st

a) $t(\alpha(g)h) = g\, t(h) g^{-1}$ "G-equivariance of t"

b) $h h' h^{-1} = \alpha(t(h)) h'$ "Peiffer identity"

Last time – we looked at cases when parts of 1)–4) were trivial.

Question: Can we build all this from H?
* We can take G trivial, but only if H is abelian. *

Note – Gauge grp of electromagnetism is $U(1)$ which is abelian.

If $H = U(1)$, this gives the "gauge 2-group" for "2-form electromagnetism".

In Yang-Mills theory, we work w/ non-abelian H.

Starting w/ nonabelian H, we want to cook up all the rest of the info.

Example: Given any group H, we can form the crossed module w/

$G = \mathrm{Aut}(H)$
$H = H$
$\alpha : G \longrightarrow \mathrm{Aut}(H)$ is the identity

We can't let t be trivial, because that forces H to be nonabelian, which it isn't.

Any grp acts on itself by conjug.

Note: Any grp elt acts as an action of the grp on itself by conjugation.

$t : H \longrightarrow G = \mathrm{Aut}(H)$ sends any $h \in H$ to the automorphism "conjugation by h."

* Must be this by Peiffer — to get

$$h h' h^{-1} = \alpha(t(h))\, h' \qquad \text{where } \alpha \text{ is ident.}$$
so need t to do above.

Then —

1) says: if $g \in \mathrm{Aut}(H)$ then "conjugation by $g(h)$" equals "conjugation by h" conjugated by g.

Electromagnetism - gauge grp $U(1)$
Y. Mills - gauge grp - any grp

conjugated by h, conj. by g.
($\underline{h}$ is conjugated by g).

$$g(h)\, h'\, g(h)^{-1} = g(h g^{-1}(h') h^{-1})$$
$$= g(h)\, g(g^{-1}(h'))\, g(h^{-1})$$
$$= g(h)\, h'\, g(h)^{-1}$$

Soon - we'll talk about 2-bundles and in particular C-2-bundles for any Lie 2-group C.
If C is the Lie 2-group built from a Lie group H in this way, people call a C-2-bundle an "H-gerbe."

(strict) <u>Lie 2-algebras</u>

Defn: A (strict) <u>Lie 2-algebra</u> is a category object in LieAlg (the category of Lie algs).

i.e. A Lie Alg of objects, Lie alg of morphisms, etc...

Reminder: Every Lie group G has a Lie alg $\mathfrak{g} = T_e G$ (tangent space to G at identity)

A <u>Lie alg</u> L is a v.space w/ operation $[\cdot,\cdot]$ that's
1) bilinear
2) antisymmetric $[x,y] = -[y,x]$
3) Jacobi: $[x,[y,z]] = [[x,y],z] + [y,[x,z]]$

Want - every Lie grp homo gives a Lie alg homo
(want to know something about morphisms!)

Every Lie grp homomorphism $F: G \to H$ gives a Lie alg. homo $dF: \mathfrak{g} \to \mathfrak{h}$
So – we get a functor:

$$\text{Diff-Grp} \longrightarrow \text{Lie Alg}$$

Hope: similarly – every Lie 2-grp gives a Lie 2-alg.

To see this: remember a Lie 2-grp is a category object in Diff-Grp, or:

$$\begin{tikzcd}
\underline{\text{Cat}} \arrow[r, "F"] \arrow[dr, dashed, "\tilde{F}"'] & \text{Diff-Grp} \arrow[d] \\
 & \text{Lie Alg}
\end{tikzcd}$$

the walking category object

· F preserves finite limits

To obtain a Lie 2-alg from our Lie 2-grp, we define a map from <u>Cat</u> to Lie Alg st the above diagram commutes.

* Need to check that functor from Diff-Grp to Lie Alg preserves limits. Then, composition, $\tilde{F}$ will preserve finite limits and will give a Lie 2-alg.

To preserve finite limits, check that it preserves

[finite products $\&$ equilizers or
terminal objects $\&$ pullbacks

- A functor that has a left adjoint preserves (finite) limits.

Just as Lie 2-grps are equivalent to Lie crossed modules,

<u>Thm</u>: Lie 2-algebras are equivalent to <u>differential crossed modules</u>, i.e.

a 4-tuples $(\underbrace{\mathfrak{g}, \mathfrak{h}}_{\text{Lie algs}}, t: \mathfrak{h} \to \mathfrak{g}$ Lie alg hom, $\underbrace{\alpha: \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})}_{\text{Lie alg. hom}})$

α was an action of a Lie grp on another, but differentiating, we get a map α above.

$\mathfrak{g}, \mathfrak{h}, t: \mathfrak{h} \to \mathfrak{g}, \alpha: \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ where

$\mathrm{Der}(\mathfrak{h})$ is the Lie alg. of "derivations" of $\mathfrak{h}$,

i.e. linear maps $f: \mathfrak{h} \to \mathfrak{h}$ st.

$$f([x,y]) = [f(x), y] + [x, f(y)] \quad \text{s.t.}$$

(differentiating a) b) from before)

a) $t(\alpha(x)y) = [x, t(y)]$ $\quad x \in \mathfrak{g}, y \in \mathfrak{h}$
b) $[y, y'] = \alpha(t(y))y'$ $\quad y, y' \in \mathfrak{h}$

Derivations

$f(gh) = f(g)f(h)$ auto, $g, h \in G$ Lie grp.

$g = e^{tx}$, $h = e^{ty}$
where $x, y \in \mathfrak{g}$

differentiating

Want to differentiate above eqn, so that it is on Lie alg $\mathfrak{g}$, involves bracket:

$$df([x,y]) = [df(x), y] + [x, df(y)]$$

* Note – the Lie alg. of $\mathrm{Aut}(G)$ is $\mathrm{Der}(\mathfrak{g})$ *

Fact: $$\frac{d}{du}\frac{d}{dv} e^{ux} e^{vy} e^{-ux} \Big|_{u,v=0} = [x,y]$$

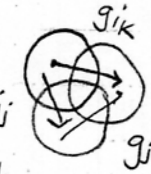
2- Bundles

Defn: A 2-bundle is a diagram $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$

where E and B are smooth categories (cat. objects in Diff) and p is a smooth functor.

Defn: Given 2 category objects C and C′ in some category, a functor (morphism) $F: C \to C'$ is a pair of morphisms
$F_0 : C_0 \to C_0'$
$F_1 : C_1 \to C_1'$

Outline — bundles
trivial bundles
locally trivial bundles
G- bundles
└ transition functions for double overlap.

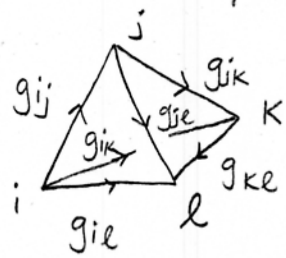


g_{ij} values in a grp.

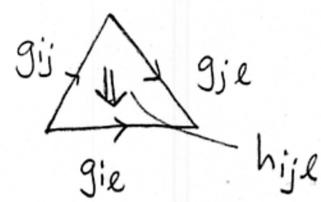
Categorifying —

2-bundles
trivial 2- bundles
locally trivial bundles
G- 2- bundles — quadruple overlaps!

tetrahedron

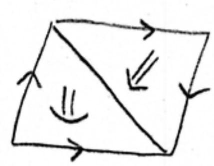


above — had g_{ij}'s equal but, here we call them isomorphic!



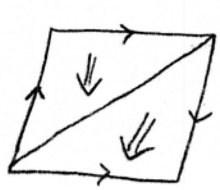
then, the h's have to satisfy an eqn coming from the tetrahedron.

Front:



=

Back:



Faces commuting, obtain an eqn.

where C_0 = object of objects of C
C_1 = object of morphisms of C

satisfying usual defn. of functor, written using commutative diagrams.

Defn: Given functors (morphisms) $F, G: C \longrightarrow C'$ between category objects, in some category, a natural transformation

$\alpha: F \Rightarrow G$, is a morphism

$$\alpha: C_0 \longrightarrow C_1'$$

satisfying usual defn. of natural transf. written using commut. diagrams.

So - we have a 2-category of:

- smooth categories
- smooth functors
- smooth natural transformations