5/28/02

<u>Adjoints</u>

$$\begin{tikzcd} \underline{\mathrm{Set}} \arrow[r, bend right, "L", "\text{``free''}"'] & \underline{\mathrm{Grp}} \arrow[l, bend right, "R", "\text{``forgetful''}"'] \end{tikzcd}$$

- given a set, we can form the free group on it.
- these maps form an adjunction because:

$$\hom(Lx, y) \cong \hom(x, Ry) \qquad \begin{array}{l} x \in \underline{\mathrm{Set}} \\ y \in \underline{\mathrm{Grp}} \end{array}$$

resembles: $\langle \psi, T\phi \rangle = \langle T^{*}\psi, \phi \rangle$ (inner products)
where T^* adjoint

<u>2- Bundles</u>

For bundles we used the category Diff:
- manifolds
- smooth maps

(but we coued have been more general)

Now- for 2-bundles we use the 2-category: Diff-Cat
(category of category objects in Diff)

- smooth categories
- smooth functors
- smooth nat transformations

Again- we coued be more general...

Defn: A 2-bundle is a smooth functor

$$\begin{tikzcd} E \arrow[d, "p"] \\ B \end{tikzcd}$$

where (for now) B is a smooth manifold

i.e. a smooth category w/ only identity morphisms.

Defn: A trivial 2-bundle w/ fiber F (smooth category) is a 2-bundle

$$\begin{tikzcd} B \times F \arrow[d, "p_1"'] \\ B \end{tikzcd}$$

ignore...

see next pg.

Defn: A 2-bundle $p: E \to B$ is trivializable if there's a commutative square:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & B' \times F \arrow[d, "p_1"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

Want $f, \tilde{f}$ to be like "isomorphisms"

* In fact – we want them to be equivalences.
(want it to be invertible up to an iso.)

... (con't) a commut. square where $\tilde{f}$ is a smooth equivalence, i.e. a smooth functor that's invertible up to specified smooth natural isomorphisms.

"take 2"

<u>Defn</u>: Given smooth categories C & D and smooth functors $F, G : C \longrightarrow D$, a <u>smooth natural isomorphism</u> $\alpha : F \Rightarrow G$ is a smooth nat. transformation w/ (unique) inverse $\alpha^{-1} : G \Rightarrow F$ st

$$\alpha\alpha^{-1} = 1_F, \quad \alpha^{-1}\alpha = 1_G$$

<u>Defn</u>: A <u>smooth equivalence</u> $F : C \longrightarrow D$ of smooth categories C and D is a smooth functor together with $G : D \to C$ and smooth natural isomorphisms

$$\alpha : FG \Rightarrow 1_C, \quad \beta : GF \Rightarrow 1_D$$

i.e. it's a quadruple (F, G, α, β)

<u>Defn</u>: A <u>morphism</u> from the 2-bundle $p : E \longrightarrow B$ to the 2-bundle $p' : E' \longrightarrow B'$ is a commutative square:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

down at bottom, B & B' are just manifolds, so f is just a smooth function.

where $\tilde{f}$ is a smooth functor and
f is a smooth map (regarded as a smooth functor).

We're in a 2-category, so we shouldn't assert that the diagram commutes strictly (equal), but up to isomorphism.

* our square commutes on the nose.*

But— we (in this case) don't have an isomorphism

$\boxed{\overset{\sim}{\Swarrow}}$

because a nat. transf. α

$$\begin{tikzcd} C \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & D \end{tikzcd}$$

assigns to any object in C a morphism in D,
but in B', we only have identity morphisms!

* We could weaken this by making the square commute <u>up to smooth natural isomorphism</u>, but it would have no effect since B' has only identity morphisms, hence the natural iso would have to be the identity.

Note: B, B' have only id. morphisms

Defn: A morphism

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"] & B' \end{tikzcd}$$

is an smooth equivalence if $\tilde{f}$ is a smooth equivalence, and f is a diffeo (regarded as a smooth equiv) (since B, B' have only id. morph).

bet. smooth categories w/ all morphisms identity

Defn: A 2-bundle is trivializable if it is equivalent to a trivial 2-bundle.

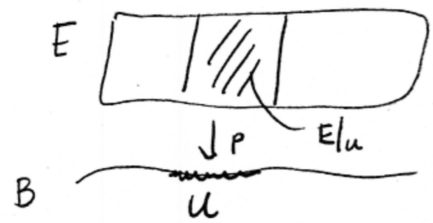
Lemma: If $p: E \longrightarrow B$ is trivializable then we can find an equivalence:

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & B \times F \arrow[d, "p_1"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$

Prop: If $p: E \longrightarrow B$ is a 2-bundle and $U \subseteq B$ is an open set, then there's a 2-bundle

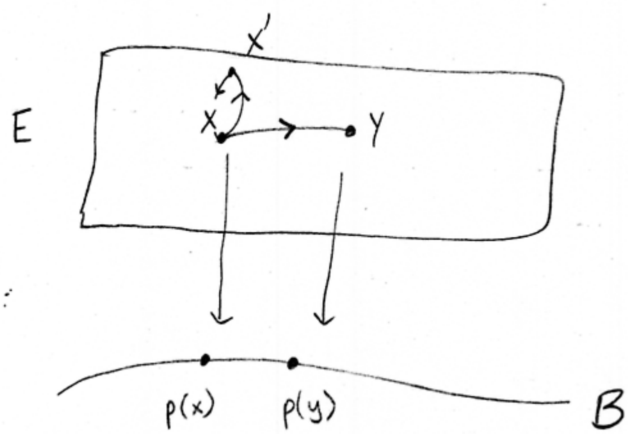
$$\begin{tikzcd} E|_U \arrow[d, "p|_{E|_U}"] \\ U \end{tikzcd}$$

where $E|_U$ is a smooth category w/



$\boxed{\text{objects}}$ $(E|_U)_0 = \{x \in E \mid p(x) \in U\}$

This picture can't exist.



* In B - only identity morphisms, so can't have a morphism bet. objects in different fibers in E.
(Each fiber is a category, but no morphisms bet. fibers.)

$\boxed{\text{morphisms}}$ $(E|_U)_1 = \{f : x \to y \text{ in } E \mid p(x) = p(y) \in U\}$

equality $p(x) = p(y)$ holds in U because only morphisms bet. objects in $U \subseteq B$ are identities.

i.e. $E|_U$ is the full subcategory of E w/ objects $\{x \in E \mid p(x) \in U\}$.

Note – p on E_1 is constant on each fiber

to prove this prop, we check:

<u>proof</u>: $\mathcal{U}$ is a manifold (open set in B)
$(E|_{\mathcal{U}})_0$ is a manifold (open set in E_0 since p_0 is cont.)

$(E|_{\mathcal{U}})_1$ is a manifold (open set in E_1 since p_1 is cont.)

$$(E|_{\mathcal{U}})_1 = p_1^{-1}(\underbrace{\{1_x \mid x \in \mathcal{U}\}}_{\text{open in } E_1})$$

and check rest of defn. ☒

<u>Remarks</u>: Given a 2-bundle $p: E \to B$ there can only be a morphism from $x \in E_0$ to $y \in E_0$ if $p(x) = p(y)$ since B has only identity morphisms.

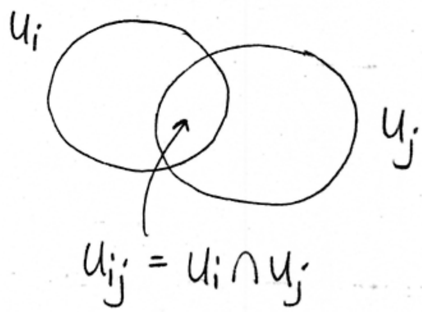
<u>Defn</u>: A 2-bundle $p: E \to B$ is <u>locally trivializable</u> if $\forall\, b \in B$ $\exists$ an open nbhd $\mathcal{U}$ containing p st

$$\begin{array}{c} E|_{\mathcal{U}} \\ \big\downarrow {\scriptstyle p|_{E|_{\mathcal{U}}}} \\ B \end{array} \quad \text{is trivializable.}$$

Suppose $p: E \to B$ is a locally trivializable 2-bundle. Pick an open cover $\mathcal{U}_i$ of B s.t. the restriction of $p: E \to B$ to $\mathcal{U}_i$ is trivializable, and pick trivializations:

$$\begin{tikzcd} E|_{U_i} \arrow[r, "t_i"] \arrow[d, "p|_{E|_{U_i}}"'] & U_i \times F_i \arrow[d, "p_i"] \\ U_i \arrow[r] & U_i \end{tikzcd}$$

Let's assume all the fibers F_i are the same, say F.
We say $p: E \to B$ is a locally trivializable 2-bundle with standard fiber F.



We can further restrict E:

$$\begin{tikzcd} U_{ij} \times F \arrow[d] & E|_{U_{ij}} \arrow[l, "t_j|_{U_{ij}}"'] \arrow[r, "t_i|_{U_{ij}}"] \arrow[d] & U_{ij} \times F \arrow[d] \\ U_{ij} & U_{ij} \arrow[l, "1"'] \arrow[r, "1"] & U_{ij} \end{tikzcd}$$

Last time — we formed α_{ij} from $U_{ij} \times F$ to itself as the composition. (involving an inverse)
But now we don't really have inverses — we have equivalences.

Recall: t_i are equivalences so we really have

$$t_i : E|_{U_{ij}} \longrightarrow U_{ij} \times F$$

$$\overline{t}_i : U_{ij} \longrightarrow E|_{U_{ij}}$$

$$\alpha : t_i \overline{t}_i \xrightarrow{\sim} 1$$

$$\beta : \overline{t}_i t_i \xrightarrow{\sim} 1$$

So we get:

$$U_{ij} \times F \xrightarrow{\overline{t}_j t_i} U_{ij} \times F$$

Note:

$$\overline{t}_j t_i (x, y) = (x, g_{ji}(x, y))$$

don't know what it is – but depends on x & y.

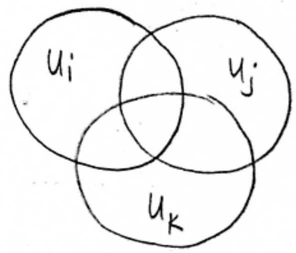
x object or morphism in U_{ij}
y object or morphism in F

where $g_{ji} : U_{ij} \times F \longrightarrow F$, is a smooth functor.
or

$$g_{ji} : U_{ij} \longrightarrow \mathrm{Aut}(F)$$

where $\mathrm{Aut}(F)$ is the category whose objects are smooth equivalences and morphisms are smooth natural isomorphisms between these.

Now — triple overlaps



$$U_{ijk} = U_i \cap U_j \cap U_k$$

Before — when talking about bundles, we had

$$g_{ij}\, g_{jk} = g_{ik} \quad (\text{restricted to } U_{ijk}),$$

but now we'll get an iso:

$$h_{ijk} : g_{ij}\, g_{jk} \overset{\sim}{\Rightarrow} g_{jk}$$

$$\begin{tikzcd}
U_{ijk} \times F \arrow[r, bend left, "\bar{t}_i t_j"] \arrow[rr, bend right, "\bar{t}_i t_k"'] & U_{ijk} \times F \arrow[r, bend left, "\bar{t}_j t_k"] \arrow[d, phantom, "(\bar{t}_i \cdot \alpha_j \cdot t_k)\Downarrow 2"] & U_{ijk} \times F \\
 & {} &
\end{tikzcd}$$

Note —
but
$h_{ijk} \neq \bar{t}_i \cdot \alpha_j \cdot t_k$

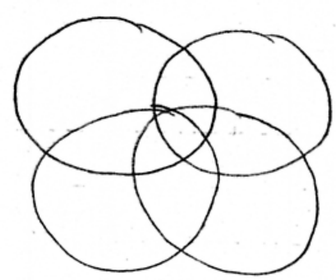
Before — we didn't have 'bars' — we had inverses so that the composite of top was <u>equal</u> to bottom. Now we have natural iso.

Or if $t_{ij} = \bar{t}_i t_j$ we get:

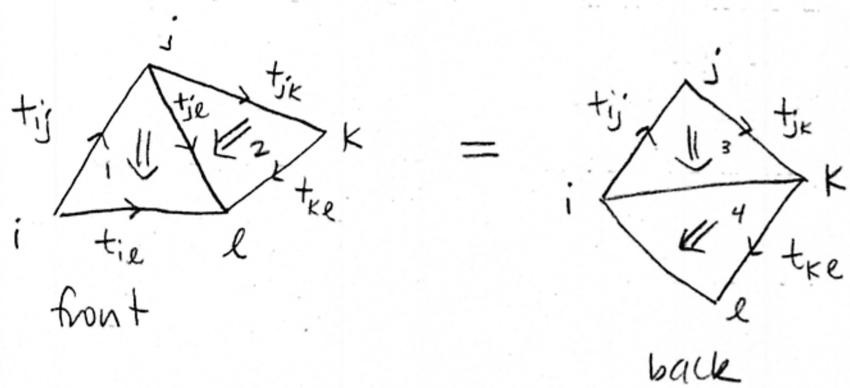
$$\begin{tikzcd} U_{ijk} \times F \arrow[r, bend left=60, "t_{ij} t_{jk}"] \arrow[r, bend right=60, "t_{ij}"'] \arrow[r, phantom, "\Downarrow t_{ijk}"] & U_{ijk} \times F \end{tikzcd}$$

$t_{ijk} = \bar{t}_i \cdot \alpha_j \cdot t_k$

Quadruple intersection:



$U_{ijkl} = U_i \cap U_j \cap U_k \cap U_l$



3 is t_{ijk}

4 is t_{ikl}

1 is t_{ijl}, 2 is t_{jkl}

We'll work this out next time.