5/30/02

2-bundles

Suppose $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$ is a locally trivializable 2-bundle, w/ standard fiber F.

Recall – B is a manifold
E is a smooth category i.e. cat object in Diff.
p is a smooth functor

Note – we can think of B as a smooth category w/ only id. morphisms

We can pick an open cover U_i of B and trivializations

arrows are functors

$$\begin{tikzcd} E|_{U_i} \arrow[r, "t_i"] \arrow[d] & U_i \times F \arrow[d, "p_1"] \\ U_i \arrow[r, "1"] & U_i \end{tikzcd}$$

"standard fiber"

Assumption: F_i are all the same – call them F.

where t_i are smooth equivalences.

Recall – our square on prev pg commutes exactly (not up to nat. iso) since $U_i \subseteq B$ has only ident. morphisms.

i.e. we specify

$$\left.\begin{aligned} t_i &: E|_{U_i} \longrightarrow U_i \times F \\ \bar{t}_i &: U_i \times F \longrightarrow E|_{U_i} \end{aligned}\right\} \text{smooth functors}$$

Then

$$\left.\begin{aligned} \alpha_i &: t_i \bar{t}_i \xrightarrow{\sim} 1_{E|_{U_i}} \\ \beta_i &: \bar{t}_i t_i \xrightarrow{\sim} 1_{U_i \times F} \end{aligned}\right\} \text{smooth nat. isomorphisms}$$

Given i, j, let $U_{ij} = U_i \cap U_j$,

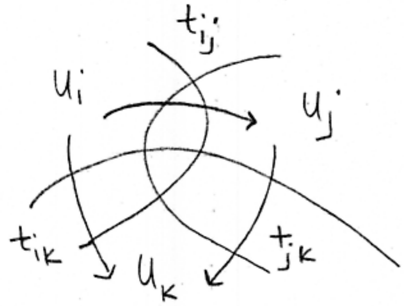
and here we have:

only weakly commutative

$$\begin{tikzcd}
 & E|_{U_{ij}} \arrow[dl, shift right, "t_i"'] \arrow[dr, "t_j"] & \\
U_{ij} \times F \arrow[ur, shift right, "\bar{t}_i"'] \arrow[rr, "t_{ij} = \bar{t}_i t_j"'] & & U_j \times F
\end{tikzcd}$$

Recall – t_i's are restrictions to a smaller set. We call t_{ij} a <u>transition functor</u>.

Given i, j, k, let $U_{ijk} = U_i \cap U_j \cap U_k$



$$\begin{tikzcd} U_{ijk} \times F \arrow[rr, "t_{ik}"] \arrow[dr, "t_{ij}"'] & & U_{ijk} \times F \\ & U_{ijk} \times F \arrow[ur, "t_{jk}"'] & \end{tikzcd}$$

non-commutative diagram (only commutes weakly – i.e. up to a smooth nat. iso.)

$\bullet = U_{ijk} \times F$

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\bar{t}_i"] \arrow[dr, "\bar{t}_i"'] & \cdot \arrow[r, bend left, "t_k"] \arrow[d, phantom, "\Uparrow t_{ijk}"] & \bullet \\ & \cdot \arrow[d, "t_j"'] & \\ & \bullet \arrow[uur, "\bar{t}_j"'] & \end{tikzcd}$$

$$\bullet \xrightarrow{\bar{t}_i} \cdot \xrightarrow{t_j} \bullet \xrightarrow{\bar{t}_j} \cdot \xrightarrow{t_k} \bullet$$

This triangle commutes weakly since we have $\alpha_j : t_j \bar{t}_j \longrightarrow 1$, so

$$\boxed{t_{ijk} = \alpha_j \text{ whiskered by } \bar{t}_i \text{ and } t_k}$$

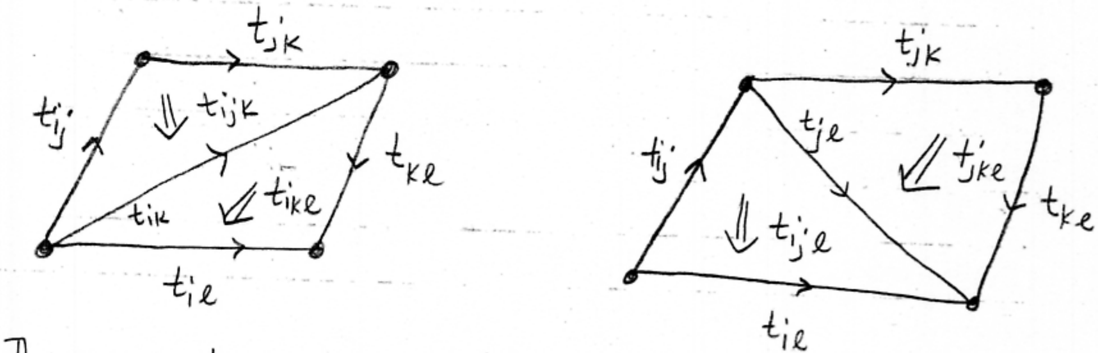
$$\begin{tikzcd} \cdot \arrow[r, "\bar{t}_i"] & \cdot \arrow[rr, "1"] \arrow[dr, "t_j"'] & \arrow[d, phantom, "\Uparrow \alpha_j"] & \cdot \arrow[r, "t_k"] & \cdot \\ & & \cdot \arrow[ur, "\bar{t}_j"'] & & \end{tikzcd}$$

so – $t_{ijk} : t_{ij} t_{jk} \xrightarrow{\sim} t_{ik}$

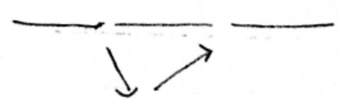
And we get that this morphism satisfies an eqn when we have a quadruple overlap!

Given i, j, k, l, let $U_{ijkl} = U_i \cap U_j \cap U_k \cap U_l$

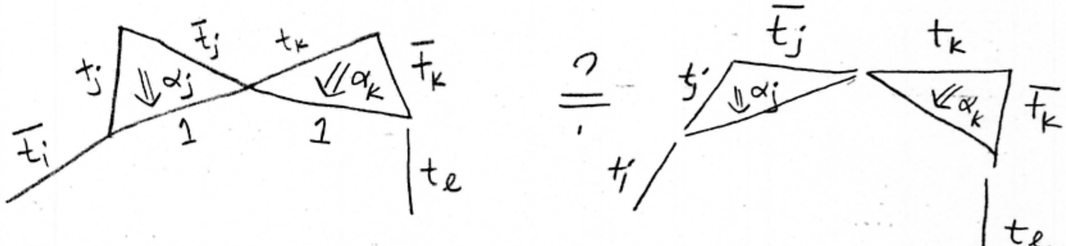
$\bullet = U_{ijkl} \times F$



These are the 2 sides of a tetrahedron.

<u>proof</u>: turn all t_{ijk}'s into  on prev pg

replace Δ by 



* This argument secretly uses the interchange law!

We say a setup (t_{ij}, t_{ijk}) satisfying this tetrahedron eq. is a Čech 2-cocycle, but with coefficients in the 2-group: Aut(F).

Aut(F) = 2 group whose objects are smooth equivalences $f: F \to F$ and whose morphisms are smooth nat. isos between these.

this is like Diff(F) from last time

Note: If F were just a smooth manifold, we'd have $\mathrm{Aut}(F) = \mathrm{Diff}(F)$ and (t_{ij}, t_{ijk}) would be a Čech 1-cocycle w/ coefficients in $\mathrm{Diff}(F)$.

(acting on fiber as a manifold)

Suppose C is a strict Lie 2-group. We say C <u>acts</u> on a smooth category F if we have a smooth functor:

$$\alpha : C \times F \longrightarrow F$$

Note - product of smooth categories is a smooth category

s.t. usual laws hold:
(this is left action)

① identity

$$\begin{tikzcd} 1 \times F \arrow[r, "i \times 1"] \arrow[d, "p_2"'] & C \times F \arrow[d, "\alpha"] \\ F \arrow[r, "1"] & F \end{tikzcd}$$

② assoc.

$$\begin{tikzcd} C \times C \times F \arrow[r, "1 \times \alpha"] \arrow[d, "m \times 1"'] & C \times F \arrow[d, "\alpha"] \\ C \times F \arrow[r, "\alpha"] & F \end{tikzcd}$$

③ inverse grp elt acts as inverse operation

$$\begin{tikzcd} C \times F \arrow[r, "\Delta \times 1"] \arrow[ddd, "p_2"'] & C \times C \times F \arrow[d, "\mathrm{inv} \times \underline{1} \times \underline{1}"] \\ & C \times C \times F \arrow[d, "1 \times \alpha"] \\ & C \times F \arrow[d, "\alpha"] \\ F \arrow[r, "\underline{1}"] & F \end{tikzcd}$$

Show this follows from ① &, ②

Prop: Given an action, we also get a homo. of 2-groups

$$C \longrightarrow \mathrm{Aut}(F)$$

Note: A weak action would be one where those diagrams only commute up to something. but we don't do this.

<u>Defn</u>: Suppose C is a Lie 2-group acting on a smooth category F. A <u>C-2-bundle</u> w/ fiber F over B is a locally trivial 2-bundle w/ fiber F over B, s.t. transition functors t_{ij} and natural transformations t_{ijk} can be chosen to "lie in C."

i.e. $t_{ij} : U_i \times F \longrightarrow U_i \times F$ gives a functor

$g_{ij} : U_i \longrightarrow \mathrm{Aut}(F)$ and we say that t_{ij} "lies in C" if

$\exists\ \tilde{g}_{ij}$ s.t.

$$\begin{tikzcd} U_i \arrow[r, "\tilde{g}_{ij}"] \arrow[dr, "g_{ij}"'] & C \arrow[d] \\ & \mathrm{Aut}(F) \end{tikzcd}$$

commutes.

Similarly for t_{ijk}.

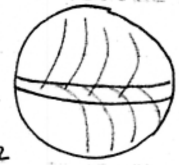
Brylinski: Loop Spaces, Characteristic Classes & Geometric Quantization

G-bundles

$G = U(1)$

U_1

$U_1 \cap U_2$



U_2

map from annulus to $U(1)$.

$U(1)$-bundles over S^2 are classified (up to iso) by integers n: the winding # of trans. funct. from annulus:

$g_{12} : U_1 \cap U_2 \longrightarrow U(1)$

C-2-bundles

- a Lie 2-grp is a Lie crossed module.

$G = 1$, $H = U(1)$, t trivial

($G = 1$, then H (top) has to be abelian by Eckman-Hilton)

$C = \{ G = 1, H = U(1), \underbrace{t, \alpha}_{\text{trivial}} \}$

This C is like G in prev. column.

C is called a "$U(1)$ gerbe"

$U(1)$-gerbes over S^3 are classified (up to equiv) by integers n.