5/7/02

<u>Defn</u>: A <u>connection</u> on a principal G-bundle $\begin{array}{c} P \\ \downarrow \\ B \end{array}$ is a family of subspaces

$$H_e \subseteq T_e P \quad \forall e \in P.$$

st

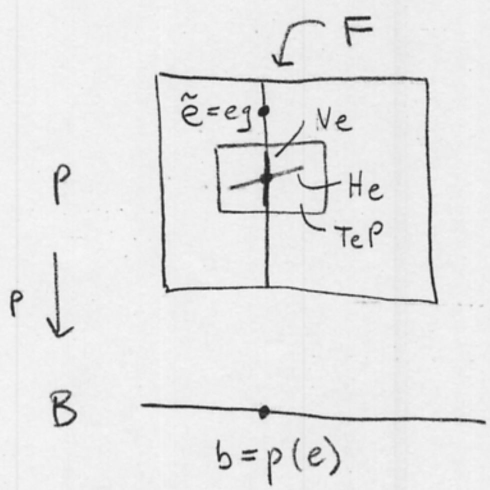
① $T_e P = V_e \oplus H_e$ where $V_e = \ker dp$ (god-given)

$dp: T_e P \to T_b B$

② H_e varies smoothly w/ e.

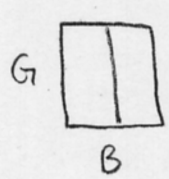
③ $H_{f(e)} = df(H_e)$ where $f: P \to P$ is given $f(e) = eg$ $\forall g \in G$.

$(G = \mathbb{R})$



Note – if we have a trivial bundle $P = B \times G$
each fiber looks exactly like G.

$$T_e P = T_b B \oplus T_g G$$



We want other ways to think about connections, esp. ways useful for calculations, which involve

$$\mathfrak{g} = T_1 G \quad \text{(the Lie alg. of grp } G\text{)}$$

In fact there is a god-given (canonical) isomorphism

$$V_e \cong \mathfrak{g} \quad \text{(lie alg)}$$

Suppose $p(e) = b \in B$. Let's call the fiber on which e lies $F = p^{-1}(b)$.

Prop: $V_e = T_e F \subseteq T_e P$

Sketch of proof: if v is tangent to F then $dp(v) = 0$.
(choose $c(t)$ w/ $c'(0) = v$ and $c(t) \in F$, then

$$p(c(t)) = b$$

$$\frac{d}{dt} p(c(t)) = 0$$

$$\quad \| $$

$$dp(c'(t)) = 0$$

or $dp(v) = 0$ so $v \in \ker dp = V_e$).

Conversely, if $v \in V_e$ then $v \in T_e F$.
So, $V_e \subseteq T_e F$, but both have dimension $= \dim F = |G|$
so $V_e = T_e F$. ∎

To get our iso $V_e \cong \mathfrak{g}$, we need $T_e F \cong \mathfrak{g}$.
The group G has a right action on P, and it maps each fiber to itself, and acts freely and transitively on F.

<u>transitively</u> means $\forall\, e, \tilde{e} \in F$, $\exists\, g \in G$ st $eg = \tilde{e}$

<u>freely</u> means $\forall\, e, \tilde{e} \in F$, $\exists$ at most one $g \in G$, $eg = \tilde{e}$.

(together — we get existence & uniqueness)

* This is because $F \cong G$ as a set w/ right G-action.

So — given $e \in F$, we get a map

$$\begin{aligned} \alpha : G &\longrightarrow F \\ g &\longmapsto eg \end{aligned}$$

$$\left.\begin{array}{l} \text{action is transitive} \Rightarrow \alpha \text{ onto} \\ \text{action is free} \Rightarrow \alpha \text{ 1-1} \end{array}\right\} \text{in fact } \alpha \text{ is a diffeo.}$$

$$\alpha : G \xrightarrow{\sim} F \quad \text{and} \quad \alpha(1) = e.$$

so

$$d\alpha : T_1 G \xrightarrow{\sim} T_e F$$

The derivative is a linear iso since α is a diffeo. But $T_1 G = \mathfrak{g}$ so we get an iso

$$\mathfrak{g} \cong T_e F.$$

Each fiber looks like G, and each tang space looks the same.

so now we'll use this iso:

$$\mathfrak{g} \cong T_eF = V_e$$

to look at things a new way:

$$\begin{tikzcd}
T_eP \arrow[d, "d\rho"'] & = & \overset{\cong \tilde{\mathfrak{g}}}{V_e} \arrow[d] & \oplus & H_e \arrow[d, "2"] \\
T_bB & = & 0 & \oplus & T_pB
\end{tikzcd}$$

But $V_e \cong \mathfrak{g}$

so tangent space = Lie alg of grp $\oplus$ horiz. subspace

Consider

$$\begin{tikzcd}
T_eP \arrow[r, "\sim"] \arrow[rr, bend right, "\omega"'] & \mathfrak{g} \oplus H_e \arrow[r, "p_1"] & \mathfrak{g}
\end{tikzcd}$$

(proj. onto 1^{st} comp)

$\mathbb{R}$ = Lie alg of $U(1)$

- ω eats tang vectors, gives Lie alg elts.
- we call f that eats tang. vectors and spits out a # is a 1-form.

so- we call ω a $\underline{\mathfrak{g}\text{-valued 1-form}}$ on P.

ie) $\forall e \in P$, $\omega_e : T_eP \longrightarrow \mathfrak{g}$ is linear.

* The kernel of $\omega_e = H_e$.

Given ω, we can recover H_e by $H_e = \ker \omega_e$.
So— we can redefine a connection in terms of ω, since ω has all the info about H_e.

<u>Thm</u>: Given a $\mathfrak{g}$-valued 1-form ω on $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ and defining $H_e = \ker \omega_e$ then H_e is a connection iff

1) restricted to $V_e \subseteq T_e P$, ω is the god-given iso $V_e \cong \mathfrak{g}$
(This implies $H_e \oplus V_e = T_e P$.]

2) ω is smooth

3) (Need group action to get along w/ ω.)

ω is invariant under the obvious action of G on $\mathfrak{g}$-valued 1-form.

Note - every grp acts on its Lie alg.

(*) $\qquad \omega_{eg} = ad(g^{-1})\omega_e \qquad e \in P,\ g \in G$

ad is a left action, we have a right action, so need to have the inverse.

where "ad" is the "adjoint action" of G on $\mathfrak{g}$:

$$\begin{aligned} G \times G &\longrightarrow G \\ (g, h) &\longmapsto ghg^{-1} \end{aligned}$$ conjugation

Note $\omega_{eg} = ad(g^{-1})\omega_e$

eats tang vectors at eg

not equal.

eats tang vectors at e

So any $g \in G$ gives a map

$$Ad(g): h \longmapsto ghg^{-1}$$

$Ad(g): \begin{matrix} G \longrightarrow G \\ \underline{1} \longmapsto \underline{1} \end{matrix}$ and differentiating we get

$$ad(g): T_1 G \longrightarrow T_1 G \text{, so}$$

$$ad(g): \mathfrak{g} \longrightarrow \mathfrak{g}$$

So we rewrite (★) from prev pg as

$$\boxed{\omega_{eg} \circ df = ad(g^{-1})\,\omega_e}$$

where $f: P \longrightarrow P$
$\quad\quad e \longmapsto eg$

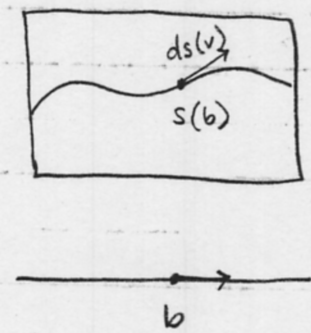
$$df: T_e P \longrightarrow T_{eg} P$$

We can turn ω into something simpler: a $\mathfrak{g}$-valued 1-form B, called "A." ("vector potential" in Yang Mills theory, which reduce to electromagnetic A (a 1-form) when $G = U(1)$ since $\mathfrak{g} \cong \mathbb{R}$.)

<u>Defn</u>: Given any bundle $\begin{array}{c} E \\ \downarrow p \\ B \end{array}$, we say a <u>section</u> is a map $s: B \to E$ st

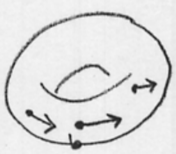
$$p(s(b)) = b.$$

$$\begin{array}{c} E \\ s \curvearrowright \downarrow p \\ B \end{array}$$



s(b) lies in the fiber over b: $p(s(b)) = b$

(depends on where b is)



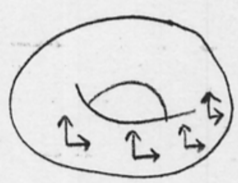
smooth v. field at each pt get a tang. vector

<u>Examples</u>: ① Given a tangent bundle $\begin{array}{c} TM \\ \downarrow \\ M \end{array}$ a section is a <u>vector field</u>.

(A v. field is a function assigning to each pt a tang vector.)

② Given a frame bundle $\begin{array}{c} FM \\ \downarrow \\ M \end{array}$ a section is a <u>frame field</u>.

· frame - basis of tang. vectors.



These are the same as trivializations of tangent bundles (which might not exist)

No frame fields on sphere by Hairy Ball Thm.

In physics – a "field" is a section of a bundle.

If we pick a section $s: B \longrightarrow P$ of our principal G-bundle $\begin{array}{c}P\\ \downarrow\\ B\end{array}$ we can turn our ω ($\mathfrak{g}$-valued 1-form on P)

into A, a $\mathfrak{g}$-val. 1-form on B, as follows:

$$A_b : T_b B \longrightarrow \mathfrak{g}$$

is given by:

$$A_b(v) = \omega(ds(v)) \quad v \in T_b B.$$

$$ds: T_b B \longrightarrow T_{s(b)} P$$

(Note – 'A' really cares only about things on the section)

In fact – we can recover ω from A since A tells us what ω does to tangent vectors at the point $s(b)$, but equivariance (condition 3) says what ω does at any other point in the same fiber. So – working w/ A, we don't lose any info instead of working w/ ω.

<u>Thm</u>: Given a section $s: B \to P$ we get a 1-1 correspondence between connections ω on P and $\mathfrak{g}$-valued 1-forms A on B.

(This <u>depends</u> on s.)

If no sections exist, pick $U \subseteq B$ st $P|_U$ is trivializable and work there!