5/27/03  Jordan Algebras

Motivation - to formally capture aspects of observables in QM in an alg.

Classical Mechanics - observables are real-valued functs.
QM - observables are self-adjoint operators on some Hilbert space here - no nice mul. as for real-valued functs, since (A,B)* ≠ A*B* unless AB = BA
However, we CAN raise the self-adj. operators to powers: A^n = A . . . A, so A^n is self-adj. when A is.

Polarization suggests a new product:
\[ 2 \cdot a \circ b = (a + b)^2 - a^2 - b^2 = (ab + ba) \]

Defn: If A is an assoc. alg. we can define a Jordan product on its underlying v. space (A,+) by giving it a new product:
\[ a \circ b = \frac{1}{2} (ab + ba) \]
Then, w/ this product (A,+,0) is a Jordan algebra.

Note: A is an alg over some field \( k \), \( \text{char}(k) \neq 2 \).

Remarks: 1) Not every Jordan alg. arises this way!
(i.e. come from an assoc. alg.)
However - practically all Jordan algs. do. arise this way.
We call such Jordan algebras special.

2) The factor of "2" is optional. We get a Jordan alg. either way, though not the same Jordan alg.
Assoc. Law: If all 3 inputs are same, we say power-associative, 2 inputs the same: alternative
\[ x (yx) = (xy)x \]

Note: if we use the "2", we get \( a \circ a = a^2 \).

**Thm:** A Jordan product is commutative, but not associative.
(Strang! We're used to losing commutativity before associativity!)
Also—a Jordan product is power-associative.

**pf:** 
**Power-assoc.** \( a \circ \ldots \circ a = a^n \)
\[ a (aa) = (aa) a \implies a \circ (a \circ a) = (a \circ a) \circ a \]

**Assoc.** \( (a \circ b) \circ c = \frac{1}{4} (abc + bac + cab + cba) \)
\[ a \circ (b \circ c) = \frac{1}{4} (abc + acb + bca + cba) \]

**ex of a power-assoc. law that doesn't come from commutativity:**
\[ a \circ ((a \circ a) \circ a) = (a \circ a) \circ (a \circ a) \]

Note: Jordan product has an identity.

**Thm:** A (special) Jordan algebra satisfies the Jordan identity:
\[ (x^2 \circ y) x = x^2 \circ (y \circ x) \]

**pf:** 
**LHS =** \( \frac{1}{4} (xxyx + yxxx + xxyy + xyyx) \)

**RHS =** \( \frac{1}{4} (xxyx + xxyy + xyyx + yxxx) \)

* Check this holds when \( x^2 = x \circ x \), but by note above—true!
At first these identities were enough:

**Defn:** A **Jordan alg** (no "special") is a vector space $(A, +)$ with a bilinear product $*$ satisfying commutativity and the Jordan identity $a^2$ has a unit. (Power-assoc. is implied) (aka a "Linear Jordan system") since by "algebra" people mean assoc.

A **special** Jordan alg. is one coming from an assoc. product in the way described. If not, we call it exceptional.

**Rmk:** The special Jordan algebras have extra identities, called $S$-identities. (infinitely many of them)

Given $x, y, z \in$ Jordan alg., we define bracket product:

$$\{xyz\} = (xoy)oz + (yoz)o(x - (zox))oy.$$  

This is used in the **Glennie identity**:

$$2\{\{x\}\{y\}\{z\}x\{y\}\{z\}x\{y\}\} - \{\{\{y\}\{x\}\{z\}x\{y\}\}z\{y\}\}z\{y\}\{z\}x\{y\}\{z\}x\{y\}\} = 0.$$  

We can't derive this from the others.

**Defn:** We call a Jordan algebra \(J\) **formally real** if \(a_j \in J\) and \(a_1^2 + \ldots + a_n^2 = 0 \Rightarrow a_j = 0 \ \forall j.\)

You can do this for any algebra. These generalize real functions in classical mechanics.
Finite dimensional formally real Jordan algebras are direct sums of simple ones:

\[ h_n = \text{n x n Hermitian matrices over } \mathbb{L} \]

1) \( h_n(\mathbb{R}), h_n(\mathbb{C}), h_n(\mathbb{H}) \quad a \cdot b = \frac{1}{2}(ab + ba) \)

2) \( h_1(\mathbb{O}) \) (same product)

3) \( \mathbb{R}^n \oplus \mathbb{R} \overset{w}{/} (X, \alpha) \cdot (Y, \beta) = (\alpha X + \beta Y, \langle X, Y \rangle + \alpha \beta) \)

The only non-special one: \( h_n(\mathbb{L}) \) is in here.

**Thm:** If a v.space \( V \) w/ Lie bracket \([\cdot, \cdot]\) (antisymmetric satisfying Jacobi id) and Jordan product \( \circ \), then these came from an assoc. alg. \( V \) w/:

\[ [x, y] = \frac{1}{2}(xy - yx), \quad x \circ y = \frac{1}{2}(xy + yx) \]

iff

1) \( (x \circ (y \circ z)) - ((x \circ y) \circ z) = [x, [y, z]] - [x, y] \circ z - [x, z] \circ y \)

2) \[ [z \circ x, y] = [x, z \circ y] + [z, x \circ y] = [x, y] \circ z - [y, z] \circ x \]

**pf:** \((\Rightarrow)\) Do it!

\((\Leftarrow)\) Define \( xy = [x, y] + x \circ y \)

Check \( (xy)z - x(yz) = 0 \)
Eight-fold Way

Recall: we grouped the 8 lightest mesons into irreps of isospin SU(2)

\[ Y = 0 \]
\[
\begin{pmatrix}
\pi^+ \\
\pi^0 \\
\pi^-
\end{pmatrix}
\]
\[ I_3 = \frac{1}{2} \]
\[ Q = I_3 + \frac{Y}{2} \]

\[ Y = 1 \]
\[
\begin{pmatrix}
K^+ \\
K^0
\end{pmatrix}
\]
\[ I_3 = \frac{1}{2} \]

\[ Y = -1 \]
\[
\begin{pmatrix}
\bar{K}^0 \\
K^-
\end{pmatrix}
\]
\[ I_3 = \frac{1}{2} \]

\[ Y = 0 \]
\[ \eta \]
\[ I_3 = 0 \]

and got this chart:
Now let's try to think of these 8 mesons as a basis for $\mathfrak{su}(3) \otimes \mathbb{C} = \mathfrak{sl}(3, \mathbb{C})$. and find self-adjoint matrices

$I_3, Y \in \mathfrak{sl}(3, \mathbb{C})$

so our 8 mesons are eigenvalues of the operators

$$[I_3, \cdot] \in \mathfrak{e}_1, [Y, \cdot]$$

whose eigenvalues are listed in the chart.

E.g. we know $\pi^+$ has $I_3 = 1, Y = 0$, so we want to find a $3 \times 3$ matrix $\pi^+ \in \mathfrak{sl}(3, \mathbb{C})$ w/ 

$$[I_3, \pi^+] = 1 \pi^+$$

$$[Y, \pi^+] = 0 \pi^+$$

Since $[I_3, \cdot]$ and $[Y, \cdot]$ have a basis of simultaneous eigenvectors, they must commute.

This will happen if $I_3$ and $Y$ commute, by the Jacobi identity:

$$[[I_3, Y], X] = [I_3, [Y, X]] - [Y, [I_3, X]]$$

If this is zero $\forall x$, then so is the LHS.

In fact, this is an iff but let's pick $I_3$ and $Y$ so they commute.
In Heisenberg's theory, we had \( I_3 \in \text{se}(2, \mathbb{C}) \) equal to
\[
I_3 = \frac{1}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]
so now let's try
\[
I_3 = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
1\#-1 + 0 = no degeneracy of eigenvalues

This means \( Y \) has to be diagonal:
\[
Y = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{pmatrix}
\]

and \( Y \in \text{se}(3, \mathbb{C}) \Rightarrow \text{tr}(Y) = a + b + c = 0 \)

Now let's find \( \pi^+ \in \text{se}(3, \mathbb{C}) \) with 
\[
[I_3, \pi^+] = 1 \pi^+
\]
and 
\[ [Y, \pi^+] = 0 \pi^+ \]

In Heisenberg's \( \text{se}(2, \mathbb{C}) \) theory:
\[
\pi^+ = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad \pi^0 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \pi^- = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

Now we want \( \pi^+ \in \text{se}(3, \mathbb{C}) \) so try
\[
\pi^+ = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
Check:

\[
[I_3, \pi^+] = \begin{bmatrix}
   (1 & 0 & 0) \\
   0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
   0 & 1 & 0 \\
   0 & 0 & 0
\end{bmatrix} = 1 \begin{bmatrix}
   0 & 1 & 0 \\
   0 & 0 & 0
\end{bmatrix} \checkmark
\]

Now want

\[
[y, \pi^+] = \begin{bmatrix}
   (a & b & c) \\
   0 & 0 & 0
\end{bmatrix} = \begin{bmatrix}
   0 & a-b & 0 \\
   0 & 0 & 0
\end{bmatrix} = (a-b) \pi^+
\]

This will be zero times \( \pi^+ \) if \( a=b \), so

\[
y = \begin{pmatrix}
   a \\
   a \\
   -2a
\end{pmatrix}
\]

We can also try

\[
\pi^- = \begin{bmatrix}
   0 & 0 & 0 \\
   1 & 0 & 0 \\
   0 & 0 & 0
\end{bmatrix}
\]

As before we have

\[
[I_3, \pi^-] = -\pi^-
\]

and also

\[
[y, \pi^-] = \begin{bmatrix}
   (a & a & -2a) \\
   0 & 0 & 0
\end{bmatrix} = (a-a) = 0 \pi^-
\]

Finally

\[
\pi^0 = \begin{pmatrix}
   1 & 0 & 0 \\
   0 & -1 & 0 \\
   0 & 0 & 0
\end{pmatrix}
\]

up to normalization

has \([I_3, \pi^0] = 0 \) since \( \pi^0 = 2I_3 \).
\[
\begin{bmatrix}
Y, \Pi^0
\end{bmatrix} = \begin{bmatrix}
(a_{a_{-2a}}, (1 - 1, 0))
\end{bmatrix} = 0 \text{ since they're diagonal}
\]

Now try a 3x3 matrix like
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
\begin{bmatrix}
I_3, (0 0 0)
\end{bmatrix} = \begin{bmatrix}
1/2 (1 - 1, 0), (0 0 0)
\end{bmatrix} = 1/2 (0 0 0)
\]

so, if it works at all, it must be \( K^+ \) or \( \bar{K^0} \).

\[
\begin{bmatrix}
Y, (0 0 0)
\end{bmatrix} = \begin{bmatrix}
(a_{a_{-2a}}, (0 0 0))
\end{bmatrix} = \begin{bmatrix}
0 0 0 \\
0 3a 0
\end{bmatrix}
\]

\[
= 3a (0 0 0)
\]

so if this is going to work, we need to have \(-3a = \pm 1\) (Y of \( K^+ \) or \( \bar{K^0} \)).

In fact, either choice works just as well, so take the conventional choice:

\[
\bar{K^0} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\text{ and } -3a = -1, \text{ so } a = 1/3
\]

So,
\[
Y = \begin{pmatrix}
1/3 \\
1/3 \\
-2/3
\end{pmatrix}
\]
Transposing gives antiparticles in Heisenberg's theory, so try

\[ K^0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

Indeed:

\[ [I_3, K^0] = -\frac{1}{2} K^0 \]

\[ [Y, K^0] = 1 \cdot K^0 \]

as our chart says:

Now try:

\[ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

Note:

\[ \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

\[ I_3 \]

so must be a particle w/ \[ I_3 = \frac{1}{2} \]

\[ \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]
so, \( Y = 1 \) for this particle, so it's the \( K^+ \)!

So, we guess \( K^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \)

\[
[I_3, K^+]^\text{Tr} = -[I_3, K^+]^{\text{Tr}}
\]

\[
(K^+)^\text{Tr} = -[I_3, K^-]
\]

So, \([I_3, K^-] = -K^-\)

i.e. transposing multiplies the eigenvalue of \([I_3, \cdot]\)

by -1.

Similarly, for \( Y \), so our guess \( K^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \) is correct.

(it's the antiparticle of \( K^+ \))

What about the \( \eta \)? Need \( \eta \in \text{sl}(3, \mathbb{C}) \) s.t.

\[
[I_3, \eta] = 0 \eta
\]

\[
[Y, \eta] = 0 \eta
\]

so \( \eta = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) and \( \eta \in \text{sl}(3, \mathbb{C}) \) implies

\[
a + b + c = 0
\]

\[
\eta = \begin{pmatrix} a & b \\ b & -a - b \end{pmatrix}
\]
We also want $\eta$ to be orthogonal to $\pi^0$.

\[(\text{amplitude for finding a } \pi^0 \text{ to be an } \eta \text{ is zero (no time)})\]

\[\langle \eta, \pi^0 \rangle = \langle \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \rangle\]

\[= tr \left( \begin{pmatrix} a \\ b \\ -a-b \end{pmatrix}^* \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)\]

\[= tr \left( \begin{pmatrix} \bar{a} \\ \bar{b} \\ -\bar{a} - \bar{b} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right)\]

\[= \bar{a} - \bar{b}\]

So we need $a = b$.

\[\eta = \begin{pmatrix} a \\ a \\ -2a \end{pmatrix}\]

Want an orthonormal basis, so

\[1 = \langle \eta, \eta \rangle = tr \left( \begin{pmatrix} a \\ a \\ -2a \end{pmatrix}^* \begin{pmatrix} a \\ a \\ -2a \end{pmatrix} \right)\]

\[= \bar{a}a + \bar{a}a + 4\bar{a}a\]

\[= 6|a|^2\]
so we know $|a| = \frac{1}{\sqrt{6}}$ so by convention we choose phase $s^+$.

$$
\eta: \left( \begin{array}{cc}
-\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
0 & \frac{2}{\sqrt{6}}
\end{array} \right)
$$