Elementary Particles

John C. Baez, April 29 2003

To specify a quantum field theory on 3+1-dimensional Minkowski spacetime, we first need to list the **elementary particles**. These are irreps of

$$\mathrm{ISpin}(3,1) \times G$$

where $\mathrm{ISpin}(3,1)$ is the universal cover of the identity component of the Poincaré group, while G is some compact Lie group called the **internal symmetry group**. Then we need to list the **interactions**, which are intertwining operators between tensor products of these irreps.

The Standard Model is the currently best accepted theory of elementary particles and their interactions, taking quantum theory and special relativity into account, but not general relativity — i.e., not gravity. There is a huge amount of experimental evidence for this theory, but its mathematical structure remains complicated and mysterious. Nobody really knows 'why' nature likes to work this way! However, the Standard Model displays many tantalizing patterns, which are probably important clues. In the following homework you will ponder the elementary particles of the Standard Model. We will not discuss the interactions yet.

In the Standard Model, the internal symmetry group is

$$G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1).$$

An irrep of a product of a bunch of groups is the same as a tensor product of irreps of these groups. Thus, to specify an elementary particle in the Standard Model, we just need to specify irreps of $\mathrm{ISpin}(3,1)$, $\mathrm{SU}(3)$, $\mathrm{SU}(2)$ and $\mathrm{U}(1)$.

The groups $\mathrm{SU}(3)$, $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ roughly correspond to the three forces other than gravity: strong, weak, and electromagnetic. The strong force holds quarks together in particles like protons and neutrons. The weak force lets one sort of quark or lepton turn into another, which is the process responsible for many radioactive decays in nuclei, and which also occurs in nuclear fusion. The electromagnetic force is the most familiar of the lot.

However, there's a sneaky twist. Specifying an irrep of $\mathrm{SU}(3)$ says how an elementary particle interacts with the strong force. Specifying an irrep of $\mathrm{U}(1)$ says how it interacts with the electromagnetic force. In particular, the unitary irreps of $\mathrm{U}(1)$ are labelled by an integer called **electric charge**: for each integer Q there is an irrep of $\mathrm{U}(1)$ on $\mathbb{C}$ in which the element $\alpha \in \mathrm{U}(1)$ acts on a vector $x \in \mathbb{C}$ to give $\alpha^Q x$, and we say this irrep describes an elementary particle of charge Q. *BUT*, the $\mathrm{U}(1)$ group describing electromagnetism is *not* the obvious $\mathrm{U}(1)$ subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1)$! It's actually some *other* $\mathrm{U}(1)$ subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1)$, which we will describe later. What this means is that the weak and electromagnetic forces are described using $\mathrm{SU}(2) \times \mathrm{U}(1)$ in a tangled-up, 'unified' way. For this reason, physicists say that specifying an irrep of $\mathrm{SU}(2) \times \mathrm{U}(1)$ says how an elementary particle interacts with the 'electroweak force'.

It would take quite a while to explain this better, and it's impossible without describing the interactions. For now, let's just see which irreps of

$$\mathrm{ISpin}(3,1) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$$

correspond to elementary particles in the Standard Model.

Again, to get an irrep of this group, we just need to take irreps of $\mathrm{ISpin}(3,1)$, $\mathrm{SU}(3)$, $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ and tensor them all together. The following chart lists what these irreps are for every particle in the Standard Model:

type of particle	ISpin(3,1) irrep	SU(3) irrep	SU(2) irrep	U(1) irrep
GAUGE BOSONS				
• gluons (SU(3) force carriers):				
$(g_{rg}, g_{rb}, g_{gr}, g_{gb}, g_{br}, g_{bg}, g_{rr} - g_{bb}, g_{bb} - g_{gg})$	massless spin-1	$\mathfrak{su}(3)$	$\mathbb{R}$	$\mathbb{R}$
• SU(2) force carriers:				
(W_1, W_2, W_3)	massless spin-1	$\mathbb{R}$	$\mathfrak{su}(2)$	$\mathbb{R}$
• U(1) force carrier:				
(W_0)	massless spin-1	$\mathbb{R}$	$\mathbb{R}$	$\mathfrak{u}(1)$
HIGGS BOSON				
• Higgs:				
(H^+, H^0)	massless spin-0	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}_1$
and its antiparticle!				
FIRST GENERATION FERMIONS				
Leptons:				
• left-handed electron neutrino and electron:				
(ν_e^L, e^L)	left-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}_{-1}$
• right-handed electron neutrino:				
(ν_e^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}_0$
• right-handed electron:				
(e^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}_{-2}$
and their antiparticles!				
Quarks:				
• left-handed up and down quarks:				
$(u_r^L, u_g^L, u_b^L, d_r^L, d_g^L, d_b^L)$	left-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}_{\frac{1}{3}}$
• right-handed up quark:				
(u_r^R, u_g^R, u_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}_{\frac{4}{3}}$
• right-handed down quark				
(d_r^R, d_g^R, d_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}_{-\frac{2}{3}}$
and their antiparticles!				
SECOND GENERATION FERMIONS				
Leptons:				
• left-handed mu neutrino and muon:				
(ν_μ^L, μ^L)	left-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}_{-1}$
• right-handed mu neutrino:				
(ν_μ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}_0$
• right-handed muon:				
(μ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}_{-2}$
and their antiparticles!				
Quarks:				
• left-handed charm and strange quarks:				
$(c_r^L, c_g^L, c_b^L, s_r^L, s_g^L, s_b^L)$	left-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}_{\frac{1}{3}}$
• right-handed charm quark:				
(c_r^R, c_g^R, c_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}_{\frac{4}{3}}$
• right-handed strange quark				
(s_r^R, s_g^R, s_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}_{-\frac{2}{3}}$
and their antiparticles!				

THIRD GENERATION FERMIONS				
Leptons:				
• left-handed tau neutrino and tau: (ν_τ^L, τ^L)	left-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}^2$	$\mathbb{C}_{-1}$
• right-handed tau neutrino: (ν_τ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}_0$
• right-handed tau: (τ^R)	right-handed massless spin-1/2	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}_{-2}$
and their antiparticles!				
Quarks:				
• left-handed top and bottom quarks: $(t_r^L, t_g^L, t_b^L, b_r^L, b_g^L, b_b^L)$	left-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}^2$	$\mathbb{C}_{\frac{1}{3}}$
• right-handed top quark: (t_r^R, t_g^R, t_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}_{\frac{4}{3}}$
• right-handed bottom quark (b_r^R, b_g^R, b_b^R)	right-handed massless spin-1/2	$\mathbb{C}^3$	$\mathbb{C}$	$\mathbb{C}_{-\frac{2}{3}}$
and their antiparticles!				

Here are some conventions used in the above chart:

- When we speak of the 'massless spin-1 irrep' of ISpin$(3,1)$, we are lying slightly. This representation is not irreducible: it is the direct sum of two irreps, the left-handed and right-handed massless spin-1 irreps. To be honest, we would thus have to double the number of rows in the section on gauge bosons. Both left-handed and right-handed gauge bosons are the same as representations of $G = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.

- We use $\mathbb{C}$ to stand for the trivial 1-dimensional complex representation of SU(3) or SU(2). We use $\mathbb{R}$ to stand for the trivial 1-dimensional real representation of SU(3), SU(2) or U(1).

- We use $\mathbb{C}^3$ as an abbreviation for the **defining** representation of SU(3): that is, its representation on $\mathbb{C}^3$ coming from the fact that SU(3) consists of 3×3 matrices. Similarly, we use $\mathbb{C}^2$ as an abbreviation for the defining representation of SU(2).

- For each irrep of ISpin$(3,1) \times \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ we list the usual names for the standard basis of this irrep. In particular, we use funny phrases like 'left-handed electron neutrino' and 'left-handed electron', 'left-handed up quark' and 'left-handed down quark', and so on as names for the standard basis of the defining representation of $\mathbb{C}^2$. For example, 'left-handed electron neutrino' is just a weird way of talking about the basis vector

  $$\nu_e^L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{C}^2,$$

  while 'left-handed electron' is a weird way of talking about

  $$e^L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{C}^2$$

  In old-fashioned physics these are considered different particles, but in the Standard Model they are really just two basis vectors for the same irrep of SU(2)!

Similarly, we use 'red', 'blue' and 'green' — or r, g, b for short — as cute names for a basis of the defining representation of SU(3) on $\mathbb{C}^3$:

$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We sometimes combine the two previous conventions. For example, the left-handed up and down quarks correspond to the representation $\mathbb{C}^3 \otimes \mathbb{C}^2$ of $\mathrm{SU}(3) \times \mathrm{SU}(2)$. This is a 6-dimensional representation with basis

$$(u^L \otimes r, u^L \otimes g, u^L \otimes b, d^L \otimes r, d^L \otimes g, d^L \otimes b),$$

which physicists abbreviate as

$$(u^L_r, u^L_g, u^L_b, d^L_r, d^L_g, d^L_b).$$

The conventions get applied in an even more complicated way for gluons, but you don't need to worry about that now.

- We describe a unitary irrep of the obvious U(1) subgroup of $\mathrm{SU}(2) \times \mathrm{U}(1)$ not by an integer, but by an integer multiple of $\frac{1}{3}$, which is called **hypercharge** and denoted y. We use $\mathbb{C}_y$ to stand for the representation of U(1) on $\mathbb{C}$ in which the element $\alpha \in \mathrm{U}(1)$ acts on $x \in \mathbb{C}$ to give $\alpha^{3y} x$.

  The reason for using multiples of $\frac{1}{3}$ is purely historical: quarks were discovered after electrons, and the electric charges of quarks come out as integer multiples of $\frac{1}{3}$ in units where the electron charge is -1. Hypercharge is not the same as electric charge, but they're closely related, so hypercharges wind up being described as multiples of $\frac{1}{3}$. (The electron, in turn, is conventionally assigned a negative electric charge because Benjamin Franklin was mixed up about which way the current flowed in lightning!)

- Every Lie group has a god-given representation on its own Lie algebra called the **adjoint representation**, obtained by differentiating the actionn of the Lie group on itself via conjugation. We use $\mathfrak{su}(3)$, $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ as an abbreviation for the adjoint representations of $\mathrm{SU}(3), \mathrm{SU}(2)$ and U(1), respectively.

- The representations of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ are representations on real vector spaces for the gauge bosons, but on complex vector spaces for the Higgs boson and fermions.

  For any complex vector space V there is a **conjugate vector space** $\overline{V}$, which we get by making V into a complex vector space in a different way, such that multiplication by i now acts as what used to be multiplication by $-i$. Similarly, for any complex representation of a group on V we get a representation of this group on $\overline{V}$, called the **conjugate representation**.

  In physics, conjugate representations describe antiparticles. In the above chart, wherever we write 'and its antiparticle', we mean that in addition to the particle corresponding to the representation shown, there is also a particle corresponding to the complex conjugate of this representation.

Now for some problems:

1. List the nine most interesting patterns you can find in the above chart — not counting the patterns mentioned in the following exercises.

2. Ignoring $\mathrm{ISpin}(3,1)$ for a moment, all the fermions of the first generation correspond to complex irreps of $G = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. Let's call the direct sum of all these irreps the **first-generation fermion representation** of G, or $\mathbf{F}$ for short.

What is the dimension of $\mathbf{F}$? If you know about Clifford algebras and spinors, what does this instantly make you wonder? If you were trying to describe $\mathbf{F}$ using the normed division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $\mathbb{O}$, what might you try?

3. $\mathbf{F}$ contains a one-dimensional trivial representation of G as a subrepresentation, so we have

$$\mathbf{F} \cong \mathbf{F}' \oplus \mathbb{C}$$

where $\mathbb{C}$ stands for this trivial representation.

What particle corresponds to this trivial representation? This particle don't interact at all with the strong, weak or electromagnetic forces, so it's very hard to see. Physicists have only become convinced of its existence in the last couple of years. Before this, they used a different version of the Standard Model, where instead of $\mathbf{F}$ they thought the first-generation fermion representation was $\mathbf{F}'$. What is the dimension of $\mathbf{F}'$? How would this make you feel if you really liked Clifford algebras and/or normed division algebras?

4. If we define second-generation and third-generation fermion representations as we did for the first generation, what are these representations like? The direct sum of all three of these representations could be called the **fermion representation** of G. What is the dimension of this representation?

5. Again ignoring $\mathrm{ISpin}(3,1)$, all the gauge bosons correspond to real irreps of G. Let's call the direct sum of all these irreps the **gauge boson representation** of G, or $\mathbf{G}$. What would mathematicians call this representation? What is the dimension of this representation?

6. Again ignoring $\mathrm{ISpin}(3,1)$, the Higgs boson corresponds to a complex irrep of $G = \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)$. Let's call this irrep the **Higgs representation** of G, or $\mathbf{H}$. What is the dimension of this representation?

7. Can you think of any interesting patterns involving $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$? It would be especially cool if this suggested a way to combine $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ into some mathematically interesting structure.

Hints: $\mathbf{G}$ is a real vector space, while $\mathbf{F}$ and $\mathbf{H}$ are complex vector spaces. Thinking about the dimensions of $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ may be helpful, especially if you're into numerology. You can think of them all as real vector spaces, but then the dimensions of $\mathbf{F}$ and $\mathbf{H}$ double. You might want to keep the $\mathrm{ISpin}(3,1)$ irreps in mind when trying to combine $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ into something interesting. You may or may not want to bring all three generations into the picture. Finally, don't forget the fact that $\mathbf{F}$ and $\mathbf{H}$ come along with their conjugate reps $\overline{\mathbf{F}}$ and $\overline{\mathbf{H}}$, corresponding to antiparticles.

There are some fairly interesting solutions to Problem 7, but I'll be satisfied with any honest attempt. A really good solution could earn you a Nobel prize!