John Baez <baez@galaxy.ucr.edu> wrote:
>By the way, there are also darn good reasons why people don't talk
>about algebras over noncommutative rings, so you'll find that defining
>an "algebra over the quaternions" is a tricky business. There may be
>some interesting way to do it, but I don't know what it is.

Define a C* algebra A over the quaternions $\mathbf{H}$ as follows: First, A should be an algebra in the usual sense over the reals $\mathbf{R}$, and A should be a left and right vector space over $\mathbf{H}$, where the different senses of scalar multiplication by a real agree. Then extend associativity by $h(ab) = (ha)b$, $(ah)b = a(hb)$, $(ab)h = a(bh)$, and $(ha)i = h(ai)$, where (as always here) $a, b \in A$ and $h, i \in \mathbf{H}$. Let * be an involution on A, with $(a+b)^* = a^* + b^*$, $(ab)^* = b^* a^*$, $(ha)^* = a^* \overline{h}$, and $(ah)^* = \overline{h} a^*$, where $\overline{h}$ for $h \in \mathbf{H}$ is the standard conjugation. Let $\|\cdot\|$ be a function from A to the nonnegative reals. Insist $\|a+b\| \leq \|a\| + \|b\|$ and the same for multiplication. If $\|a\| = 0$ implies $a = 0$, $d(a,b) := \|a - b\|$ is a metric. Insist that this is the case and that the metric is complete. Require $\|ha\| = \|h\|\|a\|$, $\|a^*\| = \|a\|$, and $\|a^* a\| = \|a\|^2$. I don't think this is always included in C* algebras, but let's say there is a multiplicative identity 1, with $h1 = 1h$ and $\|1\| = 1$. I think that's all. Then $\mathbf{H}$ itself is a C* algebra over the quaternions.

John Baez <baez@galaxy.ucr.edu> wrote:
>Ah, there's nothing like an elder sternly wagging his finger
>to get a rebellious youth eager to do what was just forbidden.
>Most of the time the youth learns the hard way why the forbidden
>was forbidden, but occasionally one is sufficiently clever to
>do something new and interesting without running into disaster.

I'm pretty sure I avoided running into disaster. Whether it's interesting depends on whether there are nontrivial examples.

>Of course this works far more generally so far. Here's what
>I hear you saying: "If we have a commutative ring K
>and a noncommutative K-algebra R, the
>category of left R-modules over K is not a monoidal category, so it
>makes no sense to define an R-algebra to be a monoid object
>in this category. So let's work instead with the category
>of R-bimodules over K.

(If R is commutative, this generalises the ordinary sort of algebra over R, which is the special case when $R = K$.)

>This *is* a monoidal category, so we can
>define an R-algebra to be a monoid object in this monoidal
>category."
>Okay, so: given a quaternionic vector space, do we get a
>quaternionic algebra of operators on this vector space?
>Of course, we get to choose what we mean by "quaternionic
>vector space" - either a left H-module or an H-bimodule.

Let V be an Rbimodule which respects the structure of R as an algebra over K. (What this means is that scalar multiplication by members of K is commutative.) Then let $L_l(V)$ be all maps $T\colon V \to V$ (written on the left) such that $T(x+y) = Tx + Ty$ and $T(xh) = (Tx)h$. Let $(T+U)x$ be $Tx + Ux$, $(hT)x := h(Tx)$, $(Th)x := T(hx)$, and $(TU)x := T(Ux)$. (Note that the left vector space structure of V is used to define both vector space structures of $L_l(V)$, while the right vector space structure of V is used to define membership in $L_l(V)$. Of course, if I wrote the operators on the right, it would be the other way around.) Then $L_l(V)$ easily satisfies the requirements of an algebra over R. If you write the operators on the right, you get a different algebra, so we must distinguish $L_l(V)$ and $L_r(V)$. If you define the opposites of an algebra and vector space in the obvious way, remembering to reverse the order of scalar multiplication as well, then $L_r(V)$ is the opposite of $L_l(V^\circ)$.

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>And: given a quaternionic Hilbert space, do we get a
>quaternionic C*-algebra of operators on this Hilbert
>space?
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Well, what's a quaternionic Hilbert space? We see above that it should be a two sided vector space V. There should be a $\mathbf{Z}$bilinear inner product $\langle,\rangle$, where $\langle xh, y\rangle = \overline{h}\langle x, y\rangle$, $\langle hx, y\rangle = \langle x, \overline{h}y\rangle$, and $\langle x, yh\rangle = \langle x, y\rangle h$. (If you think of $\langle x, y\rangle$ as x^*y, this makes sense.) Also, we want $\langle y, x\rangle = \overline{\langle x, y\rangle}$, $\langle x, x\rangle$ a nonnegative real for each $x \in V$, and $\langle x, x\rangle = 0$ only if $x = 0$. Now prove the CBS inequality:

$$\begin{aligned} 0 \leq \langle x+yh, x+yh\rangle &= \langle x, x\rangle + \langle x, yh\rangle + \langle yh, x\rangle + \langle yh, yh\rangle \\ &= \langle x, x\rangle + \langle x, y\rangle h + \overline{h}\langle y, x\rangle + \overline{h}\langle y, y\rangle h = \langle x, x\rangle + 2\Re\langle x, y\rangle h + \langle y, y\rangle \overline{h}h. \end{aligned}$$

Express $\langle x, y\rangle$ as ru, where $r \in \mathbf{R}$ and $u \in \mathbf{H}$ with $|u| = 1$. The above inequality works for any h, including $h = u^{-1}s$ for $s \in \mathbf{R}$. Then $0 \leq \langle x, x\rangle + 2rs + \langle y, y\rangle s^2$. This is a real quadratic equation in s which has at most one root, so the discriminant $4r^2 - 4\langle x, x\rangle\langle y, y\rangle \leq 0$. This yields the CBS inequality, $|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$. Then we have that $d(x, y) := \|x - y\| := \sqrt{\langle x-y, x-y\rangle}$ is a metric. This gives another condition: this metric must be complete. The definition of a Hilbert space is then finished. Note that $\|xh\| = \|x\||h|$ because $\langle xh, xh\rangle = \overline{h}\langle x, x\rangle h = \langle x, x\rangle\overline{h}h$, and $\|hx\| = |h|\|x\|$ because $\langle hx, hx\rangle = \langle \overline{h}hx, x\rangle = \langle x\overline{h}h, x\rangle = \overline{h}h\langle x, x\rangle$, where I've used the commutativity of scalar multiplication by the real $\overline{h}h$. Note that $\mathbf{H}$ is itself a quaternionic Hilbert space with $\langle h, i\rangle := \overline{h}i$.

OK, let's prove the equivalence of boundedness and continuity. If T is a linear map $V \to W$ (so $T(xh) = (Tx)h$), let T be bounded iff, for some $r \geq 0$, $\|Tx\| \leq r\|x\|$ always. Now, if T is continuous, then T is continuous at 0, so $T^{-1}\{w \in W : \|w\| \leq 1\}$ contains an open ball around $0 \in V$. That is, for some $\delta > 0$, $\|x\| < \delta$ implies $\|Tx\| \leq 1$. If $x \in V$ and $\epsilon > 0$, then $\|x\delta/(\|x\| + \epsilon)\| = \|x\|\delta/(\|x\| + \epsilon) < \delta$, so $1 \geq \|T(x\delta/(\|x\| + \epsilon))\| = \|Tx\|\delta/(\|x\| + \epsilon)$, or $\|Tx\| \leq (\|x\| + \epsilon)/\delta$. As $\epsilon \to 0$, we see that $1/\delta$ is the r required for T to be bounded. Conversely, if T is bounded by r, $T^{-1}\{w \in W : \|w\| < \zeta\}$ contains an open ball around $0 \in V$ of radius ζ/r. This means T is continuous at 0, and continuity can be moved from point to point by $\mathbf{Z}$linearity.

We can define $\|T\|$ as $\inf\{r \geq 0 : \forall x \in V, \|Tx\| \leq r\|x\|\}$ or as $\sup\{\|Tx\| : \|x\| \leq 1\}$, which is the same as $\sup\{\|Tx\| : \|x\| = 1\}$. The proof is the same as usual as long as you remember to use the correct scalar multiplication in the proof. (Actually, this is unnecessary since the scalars involved are real.)

Let $B_l(V)$ be the two sided vector space of bounded linear maps $V \to V$. The maps are written on the left and must respect right scalar multiplication but not left scalar multiplication. We need that $T, U \in B_l(V)$ implies $T + U \in B_l(V)$ and $\|T + U\| \leq \|T\| + \|U\|$; $\|(T + U)x\| = \|Tx + Ux\| \leq \|Tx\| + \|Ux\| \leq \|T\|\|x\| + \|U\|\|x\| = (\|T\| + \|U\|)\|x\|$. We need that $T, U \in B_l(V)$ implies TU in $B_l(V)$ and $\|TU\| \leq \|T\|\|U\|$; $\|(TU)x\| = \|T(Ux)\| \leq \|T\|\|Ux\| \leq \|T\|\|U\|\|x\|$. We need that $h \in H$ and $T \in B_l(V)$ imply $hT \in B_l(V)$ and $\|hT\| = |h|\|T\|$; $\|(hT)x\| = \|h(Tx)\| = |h|\|Tx\| \leq |h|\|T\|\|x\|$, so $hT \in B_l(V)$ and $\|hT\| \leq |h|\|T\|$; for some x with $\|x\| = 1$, $\|Tx\| = \|T\|$, so $\|hTx\| = |h|\|T\|$, so $\|hT\| \geq |h|\|T\|$. (Note that $\|Th\| = \|T\||h|$ will follow when we do adjoints.)

It's easier to write it than to read it. You can trust me; it all falls out easily.

OK, next come adjoints. We need the Riesz representation theorem. Let $L\colon V \to \mathbf{H}$ be linear, and let W be $\ker L$. If $W = V$, then $L = 0$ and $Lx = \langle 0, x\rangle$ for all $x \in V$. Otherwise, choose a nonzero v in $W^\perp := \{v \in V : \forall w \in W, \langle v, w\rangle = 0\}$. $Lv \neq 0$, so $L(v/Lv) = Lv/Lv = 1$. So, rename v/Lv just 'v'. If $x \in V$, then $L(x - vLx) = Lx - (Lv)Lx = 0$, so $x - vLx \in W$. $v \in W^\perp$, so $0 = \langle v, x - vLx\rangle = \langle v, x\rangle - \langle v, v\rangle Lx$, or $\langle v, x\rangle = \langle v, v\rangle Lx$. So, let L^* in V be $v/\langle v, v\rangle$; then $\langle L^*, x\rangle = \langle v/\langle v, v\rangle, x\rangle = \langle v, v\rangle^{-1}\langle v, x\rangle = Lx$. Suppose (no longer assuming $W \neq V$; if $W = V$, let L^* be 0) that $\langle l, x\rangle = Lx = \langle L^*, x\rangle$ for all $x \in V$. Then $\langle l - L^*, l - L^*\rangle = \langle l, l - L^*\rangle - \langle L^*, l - L^*\rangle = 0$, so $l = L^*$. $\langle (L+M)^*, x\rangle = (L + M)x = Lx + Mx = \langle L^*, x\rangle + \langle M^*, x\rangle = \langle L^* + M^*, x\rangle$, so $(L + M)^* = L^* + M^*$ by the same reasoning as proved $l = L^*$ above. $\langle (hL)^*, x\rangle = hLx = h\langle L^*, x\rangle = \langle L^*\overline{h}, x\rangle$, so $(hL)^* = L^*\overline{h}$. $\langle (Lh)^*, x\rangle = Lhx = \langle L^*, hx\rangle = \langle \overline{h}L^*, x\rangle$, so $(Lh)^* = \overline{h}L^*$. $L(L^*/\|L^*\|) = LL^*/\|L^*\| = \langle L^*, L^*\rangle/\|L^*\| = \|L^*\|$, so $\|L\| \geq \|L^*\|$. Also, $\|Lx\| = \|\langle L^*, x\rangle\| \leq \|L^*\|\|x\|$, so $\|L\| \leq \|L^*\|$. Thus, $\|L\| = \|L^*\|$. So, * is an antiisomorphism of the quaternionic Banach spaces $B_l(V, \mathbf{H})$ and V.

I've defined a sort of adjoint of bounded linear functionals; now let me extend this to any bounded linear operator. Suppose $T\colon V \to W$ is a bounded linear operator. For $w \in W$, define $w^*T\colon V \to \mathbf{H}$ by

$(w^*T)v = \langle w, Tv\rangle$. $\|(w^*T)v\| = \|\langle w, Tv\rangle\| \le \|w\|\|Tv\| \le \|w\|\|T\|\|v\|$, so w^*T is bounded. Define $T^\dagger\colon W \to V$ by $T^\dagger w := (w^*T)^*$. Then $\langle T^\dagger w, v\rangle = (w^*T)v = \langle w, Tv\rangle$ as desired. Note, if $v \in V$, we have $v^*T^\dagger\colon W \to \mathbf{H}\colon w \mapsto \langle v, T^\dagger w\rangle$. $\langle v, T^\dagger w\rangle = \overline{\langle T^\dagger w, v\rangle} = \overline{\langle w, Tv\rangle} = \langle Tv, w\rangle$, so $(v^*T^\dagger)^* = Tv$ and $T^{\dagger\dagger} = T$.

$$\begin{aligned}\left\langle (T+U)^\dagger w, v\right\rangle &= \langle w, (T+U)v\rangle = \langle w, Tv + Uv\rangle = \langle w, Tv\rangle + \langle w, Uv\rangle \\ &= \langle T^\dagger w, v\rangle + \langle U^\dagger w, v\rangle = \langle T^\dagger w + U^\dagger w, v\rangle = \langle (T^\dagger + U^\dagger)w, v\rangle,\end{aligned}$$

so $(T+U)^\dagger = T^\dagger + U^\dagger$. $\left\langle (hT)^\dagger w, v\right\rangle = \langle w, hTv\rangle = \langle \overline{h}w, Tv\rangle = \langle T^\dagger \overline{h} w, v\rangle$, so $(hT)^\dagger = T^\dagger \overline{h}$. $\left\langle (Th)^\dagger w, v\right\rangle = \langle w, Thv\rangle = \langle T^\dagger w, hv\rangle = \langle \overline{h} T^\dagger w, v\rangle$, so $(Th)^\dagger = \overline{h}T^\dagger$. $\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^\dagger Tx, x\rangle \le \|T^\dagger Tx\|\|x\| \le \|T^\dagger T\|\|x\|^2$, so $\|T\|^2 \le \|T^\dagger T\| \le \|T^\dagger\|\|T\|$, or $\|T\| \le \|T^\dagger\|$. But this means $\|T^\dagger\| \le \|T^{\dagger\dagger}\| = \|T\|$, so $\|T^\dagger\| = \|T\|$. So, † is an antiisomorphism from $B_l(V, W)$ to $B_l(W, V)$.

Let's check that this notion of † agrees with the previous usage of * and $^-$. If $L\colon V \to \mathbf{H}$, I defined L^* as an element of V and $L^\dagger$ as an element of $B_l(\mathbf{H}, V)$. But V and $B_l(\mathbf{H}, V)$ are naturally isomorphic (using right scalar multiplication). $\langle L^*h, x\rangle = \overline{h}\langle L^*, x\rangle = \overline{h}Lx = \langle h, Lx\rangle$, so L^* corresponds to $L^\dagger$. Also, for $x \in V \cong B_l(\mathbf{H}, V)$, $\langle\langle x, y\rangle, h\rangle = \overline{\langle x, y\rangle}h = \langle y, x\rangle h = \langle y, xh\rangle$, so $x^\dagger$ is the functional that maps y to $\langle x, y\rangle$, consistent with the 'v^*T' notation above. Finally, $\mathbf{H}$ is a subset of any $B_l(V)$; $\langle \overline{h}x, y\rangle = \langle x, hy\rangle$, so $h^\dagger = \overline{h}$. Thus, there is really only one $^- = {}^* = {}^\dagger$, even including my $\langle x, y\rangle = x^*y$ comment from long ago.

There are only a few requirements left for $B_l(V)$ to be a C* algebra. $\left\langle (TU)^\dagger x, y\right\rangle = \langle x, TUy\rangle = \langle T^\dagger x, Uy\rangle = \langle U^\dagger T^\dagger x, y\rangle$, so $(TU)^\dagger = U^\dagger T^\dagger$. $\|T^\dagger T\| = \|T\|^2$ follows from the proof above that $\|T^\dagger\| = \|T\|$. There is obviously an identity map $1\colon V \to V\colon x \mapsto x$, and $(1h)x = 1(hx) = hx = h(1x) = (h1)x$, so $1h = h1$. Therefore, $B_l(V)$ is a quaternionic C* algebra.

Of course, $B_r(V)$ is just as valid.

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>And - sort of sneaking up on the same questions from the
>other side - do n x n quaternionic matrices form a
>quaternionic C*-algebra in your sense?
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So far, the only examples I've given of quaternionic C* algebras are $\mathbf{H}$ itself and $B(V)$ for V a quaternionic Hilbert space, and the only example I've given of a quaternionic Hilbert space is $\mathbf{H}$, so the only example of a quaternionic C* algebra I've given is $\mathbf{H}$. Hopefully, there are more examples than this! We should at least consider if $\mathbf{H}^n$ is a Hilbert space.

$\mathbf{H}^n$ is easily a two sided quaternionic vector space, and multiplication by real scalars commutes. Define $\langle v, w\rangle$ to be $\sum_i \overline{v_i}w_i$. $\langle , \rangle$ is $\mathbf{Z}$bilinear. $\langle vh, w\rangle = \sum \overline{v_i h} w_i = \sum \overline{h}\overline{v_i} w_i = \overline{h}\sum \overline{v_i}w_i = \overline{h}\langle v, w\rangle$. $\langle hv, w\rangle = \sum \overline{hv_i}w_i = \sum \overline{v_i}\overline{h}w_i = \langle v, \overline{h}w\rangle$. $\langle v, wh\rangle = \sum \overline{v_i}w_i h = (\sum \overline{v_i}w_i)h = \langle v, w\rangle h$. $\langle w, v\rangle = \sum \overline{w_i}v_i = \sum \overline{\overline{v_i}w_i} = \overline{\sum \overline{v_i}w_i} = \overline{\langle v, w\rangle}$. $\langle v, v\rangle = \sum \overline{v_i}v_i = \sum |v_i|^2 \ge 0$. If $\langle v, v\rangle = 0$, then $0 = \sum |v_i|^2$, so each $|v_i|^2 = 0$, so each $v_i = 0$, so $v = 0$. The norm is the usual one on $\mathbf{R}^{4n}$, so completeness follows. Therefore, $\mathbf{H}^n$ is indeed a Hilbert space over $\mathbf{H}$, from which follows that $B_l(\mathbf{H}^n)$ is a C*algebra over $\mathbf{H}$.

Of course, this leaves the question: does $B_l(\mathbf{H}^n)$ equal $\mathrm{Mat}_n(\mathbf{H})$? Well, $\mathrm{Mat}_n(\mathbf{H})$ is clearly $L_l(\mathbf{H}^n)$, so is every matrix bounded? Yes, by the square root of the sum of the squares of the magnitudes of its elements.

Note: Some of the proofs here are adapted from John B. Conway, 1990, A Course in Functional Analysis, 2nd ed.

John Baez <baez@galaxy.ucr.edu> wrote:

```
>What I'm about to say is a bit abstract,
>so fasten your seatbelt.  There's a very nice bicategory whose
>objects are all the K-algebras in the world.  In this bicategory, the
>morphisms X: R -> R' are just the R,R'-bimodules over K.  I assume you know,
>or can guess, that an R,R'-bimodule X over K is a K-module that's both a left
>R-module and a right R'-module, satisfying the following compatibility
>condition: (r x) r' = r (x r').
>And in this bicategory, the 2-morphisms from X: R -> R' to
>X': R -> R' are just the R,R'-bimodule morphisms f: X -> X'.
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>You may remember that a
>bicategory with one object is a monoidal category. So suppose
>we take the above bicategory and chop it down until it has just one
>object. In other words: take a single K-algebra R and consider all the
>R,R-bimodules over K and all the bimodule morphisms between these. Then
>we get a monoidal category!
>So you really do have a monoidal category.
>You may think this digression was overkill, and it probably was,
>but sometimes I think what we really need to do eventually is
>consider the real numbers, the complex numbers and the quaternions
>all simultaneously as part of a single package. One way to do this
>is look at a little sub-bicategory of the above one for K = the reals, whose objects
>are just the reals, the complexes, and
>the quaternions. It's an idea worth keeping in mind.

This then generalises the concept of C*algebra over **C**, because multiplication by arbitrary complex scalars needn't commute.

>It would be fun to see if Stephen Adler did
>things the same way.

You can tensor an operator on the left (a right module homomorphism) and an operator on the right (a left module homomorphism) to form a Kmodule homomorphism (just as you tensor a right module and a left module to form a Kmodule). The space $L_K(V,W)$ of Khomomorphisms is an Rbimodule, indeed an Ralgebra when $V=W$. The only hard part in the definition is deciding whether to write $f\otimes g$ on the left or the right. If on the left, the definition is $(f\otimes g)(x\otimes y):=fx\otimes yg$; if on the right, the definition is $(x\otimes y)(f\otimes g):=fx\otimes yg$. But one really ought to say $f(x\otimes y)g:=fx\otimes yg$. This doesn't violate any spirit of commutativity and shows that Kmodule homomorphisms are naturally thought of as living on both the left and right sides of their arguments. But $\otimes$ is *not* a map $L_l(V)\otimes L_r(V)\to L_K(V)$, because $fr\otimes g\neq f\otimes rg$ in general. For similar reasons, we can't tensor an operator on the right and an operator on the left to produce a bimodule homomorphism (even though you tensor a left module and a right module to form a bimodule) because the definition $x(f\otimes g)y:=xf\otimes gy$ doesn't satisfy $xr(f\otimes g)y = x(f\otimes g)ry$, even though we would have $fr\otimes g=f\otimes rg$.

John Baez <baez@galaxy.ucr.edu> wrote:
>How does the fact that the quaternions are themselves a *-algebra
>enter into the game of opposites? Being a *-algebra, they are isomorphic to
>*their* opposite algebra, which gives a way to turn any left H-module
>into a right H-module and vice versa. I guess it also lets you
>turn any H-bimodule into a new one with right and left structures
>flipped: NEW h v h' = OLD h'* v h*
>It seems like this should be important.

Define $\overline{V}$ to be V as an Abelian group and hvi in $\overline{V}$ to be $\bar{i}v\bar{h}$ in V. Then $\overline{V}$ and V are identical as real vector spaces. If V is a Hilbert space, so is $\overline{V}$, and Riesz says $\overline{V}\cong B_l(V,\mathbf{H})$. Define $\overline{A}$ to be A° as a ring and $\overline{A}$ as a vector space. If A is a *algebra, then * is an isomorphism between A and $\overline{A}$. One may say the pair $(\mathbf{H},\overline{V})$ is isomorphic to the pair $(\mathbf{H},V^\circ)$, where $\mathbf{H}$ is also allowed to undergo an isomorphism, in this case $^-$. Similarly, the pair $(\mathbf{H},\overline{A})$ is isomoprhic to the pair $(\mathbf{H},A^\circ)$.

Of course, you can just as easily say that $\overline{V}\cong B_r(V,\mathbf{H})$. This gives a natural isomorphism between $B_l(V,\mathbf{H})$ and $B_r(V,\mathbf{H})$. In general, one has an isomorphism between $B_l(V,\mathbf{H}^n)$ and $B_r(V,\mathbf{H}^n)$, where $f\in B_l(V,\mathbf{H}^n)$ is mapped to $v\mapsto\overline{fv}$. Here, $^-$ acts on $\mathbf{H}^n$ by conjugating coordinatewise. More generally, any isomorphism * from W to $\overline{W}$ induces an isomorphism from $B_l(V,W)$ to $B_r(V,\overline{W})$. If W is a *algebra, we have such an isomorphism *. We can do the same thing with the argument in the first position.

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>Now that I think about it, I seem to remember some theory of gadgets like
>Hilbert spaces which are representations of a C*-algebra A but which
>have an "inner product" taking values in A.  I'd seen this developed
>over the complex numbers, but if one developed it over the reals, it would
>include what you're doing as a special case - except for the stuff where
>you use the fact that the quaternions are a division algebra.  In this
>theory, you certainly have to get everything to fall onto the correct
>side.
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John Baez <baez@galaxy.ucr.edu> wrote:

```
>Is there a nice way to characterize
>L_l in the category of bimodules (e.g. by some nice universal property)?
>If there's any justice in the world, L_l will be a functor:
>L_l: H-Bimod^op x H-Bimod -> H-Bimod.
>There will also be a functor: L_r: H-Bimod^op x H-Bimod -> H-Bimod.
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Well, we must define the action of the functors on morphisms. Given bimodule homomorphisms $f\colon V \to V'$ and $g\colon W \to W'$, what is $L_l(f,g)\colon L_l(V',W) \to L_l(V,W')$? Well, if $T \in L_l(V',W)$, then $gTf \in L_l(V,W')$. $g(T+U)f = gTf + gUf$, $g(hT)f = h(gTf)$, and $g(Th)f = (gTf)h$. Therefore, $L_l(f,g)\colon T \mapsto gTf$ is a morphism. For $L_r(f,g)$, the situation is the same, except that gTf is better called 'fTg' since the functions are written on the right. Given $f\colon V \to V'$, $f'\colon V' \to V''$, $g\colon W \to W'$, and $g'\colon W' \to W''$, is $L_l(f'f, g'g)$ equal to $L_l(f,g')L_l(f',g)$? Yes, because $(g'g)T(f'f) = g'(gTf')f$. Also, $L_r(f'f, g'g) = L_r(f,g')L_r(f',g)$, because $(ff')T(gg') = f(f'Tg)g'$, which is all in the correct order. Finally, if $1\colon V \to V$ and $1\colon W \to W$ are identity maps, $L_l(1,1)\colon L_l(V,W) \to L_l(V,W)$ is the identity map, since $1T1 = T$, and $L_r(1,1)\colon L_r(V,W) \to L_r(V,W)$ is the identity map, since $1T1 = T$.

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>I would be very happy if these functors had nice characterizations
>in terms of tensor: H-Bimod x H-Bimod -> H-Bimod.
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I claim $L_l(V,\cdot)$ is the right adjoint of $\cdot \otimes V$. This means there's a natural transformation ι from the identity functor to the functor $L_l(V, \cdot \otimes V)$. That is, for every object W, there's a morphism from W to $L_l(V, W \otimes V)$. Obviously, this morphism sends w to f, where $fv := w \otimes v$. Check that $f \in L_l(V, W \otimes V)$; $f(vh) = w \otimes vh = (w \otimes v)h = (fv)h$. Check that the map $w \mapsto w \otimes \cdot$ is a morphism; $hw \otimes v = h(w \otimes v) = h(fv) = (hf)v$, and $wh \otimes v = w \otimes hv = f(hv) = (fh)v$. I also need a natural transformation ϵ from the functor $L_l(V,\cdot) \otimes V$ to the identity functor, that is, for every object W, a morphism from $L_l(V,W) \otimes V$ to W. Obviously, this morphism sends $f \otimes v$ to fv for $f \in L_l(V,W)$. Check that the map $f \otimes v \mapsto fv$ is a morphism; $h(f \otimes v) = hf \otimes v \mapsto (hf)v = h(fv)$, and $(f \otimes v)h = f \otimes vh \mapsto f(vh) = (fv)h$. Check that the morphism is well defined; $fh \otimes v \mapsto (fh)v = f(hv) \leftarrow\!\shortmid f \otimes hv$. Therefore, $L_l(V,\cdot)$ is the right adjoint of $\cdot \otimes V$.

I also claim $L_r(V,\cdot)$ is the right adjoint of $V \otimes \cdot$. This means there's a natural transformation ι from the identity functor to the functor $L_r(V, V \otimes \cdot)$. That is, for every object W, there's a morphism from W to $L_r(V, V \otimes W)$. Obviously, this morphism sends w to f, where $vf = v \otimes w$. Check that $f \in L_r(V, V \otimes W)$; $(hv)f = hv \otimes w = h(v \otimes w) = h(vf)$. Check that the map $w \mapsto \cdot \otimes w$ is a morphism; $v \otimes hw = vh \otimes w = (vh)f = v(hf)$, and $v \otimes wh = (v \otimes w)h = (vf)h = v(fh)$. I also need a natural transformation ϵ from the functor $V \otimes L_r(V,\cdot)$ to the identity functor, that is, for every object W, a morphism from $V \otimes L_r(V,W)$ to W. Obviously, this morphism sends $v \otimes f$ to vf for $f \in L_r(V,W)$. Check that the map $v \otimes f \mapsto vf$ is a morphism; $h(v \otimes f) = hv \otimes f \mapsto (hv)f = h(vf)$, and $(v \otimes f)h = v \otimes fh \mapsto v(fh) = (vf)h$. Check that the morphism is well defined; $vh \otimes f \mapsto (vh)f = v(hf) \leftarrow\!\shortmid v \otimes hf$. Therefore, $L_r(V,\cdot)$ is the right adjoint of $V \otimes \cdot$.

Note that **Z**linearity needs to be checked a lot above, but I never bothered since it's always pretty obvious. There are also some commutative diagrams to check, but I didn't write them down, since I've been writing enough. I checked them.