Linear Orderings, Cyclic Orderings and Permutations

John C. Baez, April 1, 2004

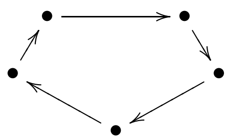
A **linear ordering** *of a set* S*, also called a* **total ordering***, is a binary relation* $<$ *on* S *that is:*

- *irreflexive* ($x \not< x$),
- *asymmetric* ($x < y \Longrightarrow y \not< x$)
- *transitive* ($x < y$ & $y < z \Longrightarrow x < z$)
- *and linear* ($x \neq y \Longrightarrow x < y$ or $y < x$).

In pictures, a linear ordering looks something like this:

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$

A 'cyclic ordering', on the other hand, looks like this:



More formally, we can define a **cyclic ordering** *of the finite set* S *to be a permutation* $\sigma\colon S \to S$ *with exactly one orbit. The permutation maps each element of* S *to the 'next one on the cycle'. We can also define a cyclic ordering to be an equivalence class of linear orderings, where the linear ordering of* $\{x_1, \ldots, x_n\}$ *with*

$$x_1 < x_2 < \cdots < x_{n-1} < x_n$$

is equivalent to the total ordering with

$$x_n < x_1 < x_2 < \cdots < x_{n-1}.$$

("And the last shall be first." — Matthew 19.) However, this definition is valid only if S *is nonempty; the empty set has no cyclic ordering (because its unique permutation has zero orbits).*

Let L *be the structure type "being a linearly ordered finite set", and let* C *be the structure type "being a cyclically ordered finite set". There are some nice relations between these two structure types.*

1. Compute the generating functions $|L|$ and $|C|$ directly, by counting the number of linear orderings and cyclic orderings on an n-element set.

2. Using 1. show that

$$\begin{aligned} \tfrac{d}{dz}|C| &= |L| \\ \tfrac{d}{dz}|L| &= |L|^2 \\ e^{|C|} &= |L|. \end{aligned}$$

3. Do the above equations between generating functions come from natural isomorphisms between the structure types? Show that

$$\frac{D}{DZ}C \cong L$$

and

$$\frac{D}{DZ}L \cong L^2$$

but

$$E^C \ncong L.$$

Hint: for the last one, let P be the structure type "being a finite set equipped with a permutation of its elements". Show that $E^C \cong P$ and $P \ncong L$.

P and L are an interesting pair of structure types. Even though a permutation is very different from a linear order:

$$P \ncong L$$

there are just as many permutations of a finite set as linear orders on it:

$$|P| = |L|$$

and we've seen above that both can be defined in terms of cyclic orderings:

$$P \cong E^C, \qquad\qquad L \cong \frac{DC}{DZ}.$$

4. Let $1//2!$ be the groupoid with one object and $\mathbb{Z}_2$ as the group of automorphisms of this object, so that

$$|1//2!\,| = 1/2.$$

Calculate the groupoid cardinality of $C(1//2!)$. This is the groupoid of 'half-colored cyclically ordered finite sets'.