Equivalence of Categories

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1. Prove that a functor $F\colon C \to D$ is an equivalence iff it is essentially surjective, full and faithful.

Hint: let me remind you of all the necessary definitions.

Definition 1. *A* **category** C *consists of:*

- *a collection* $\mathrm{Ob}(C)$ *of* **objects**.
- *for any pair of objects* x, y, *a set* $\hom(x, y)$ *of* **morphisms** *from* x *to* y. *(If* $f \in \hom(x, y)$ *we write* $f\colon x \to y$.*)*

equipped with:

- *for any object* x, *an* **identity morphism** $1_x\colon x \to x$.
- *for any pair of morphisms* $f\colon x \to y$ *and* $g\colon y \to z$, *a morphism* $fg\colon x \to z$ *called the* **composite** *of* f *and* g.

such that:

- *for any morphism* $f\colon x \to y$, *the* **left and right unit laws** *hold:* $1_x f = f = f 1_y$.
- *for any triple of morphisms* $f\colon w \to x$, $g\colon x \to y$, $h\colon y \to z$, *the* **associative law** *holds:* $(fg)h = f(gh)$.

We usually write $x \in C$ as an abbreviation for $x \in \mathrm{Ob}(C)$. An **isomorphism** is a morphism $f\colon x \to y$ with an **inverse**, i.e. a morphism $g\colon y \to x$ such that $fg = 1_x$ and $gf = 1_y$.

Definition 2. *Given categories* C, D, *a* **functor** $F\colon C \to D$ *consists of:*

- *a function* $F\colon \mathrm{Ob}(C) \to \mathrm{Ob}(D)$.
- *for any pair of objects* $x, y \in \mathrm{Ob}(C)$, *a function* $F\colon \hom(x, y) \to \hom(F(x), F(y))$.

such that:

- F **preserves identities**: *for any object* $x \in C$, $F(1_x) = 1_{F(x)}$.
- F **preserves composition**: *for any pair of morphisms* $f\colon x \to y$, $g\colon y \to z$ *in* C, $F(fg) = F(f)F(g)$.

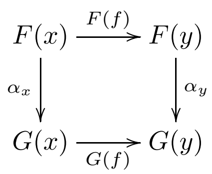
It's not hard to define identity functors and composition of functors, and to check the left and right unit law and associative law for these.

Definition 3. *Given functors* $F, G\colon C \to D$, *a* **natural transformation** $\alpha\colon F \Rightarrow G$ *consists of:*

- *a function* α *mapping each object* $x \in C$ *to a morphism* $\alpha_x\colon F(x) \to G(x)$

such that:

- *for any morphism* $f\colon x \to y$ *in* C, *this diagram commutes:*



With a little thought you can figure out how to compose natural transformations $\alpha: F \to G$ and $\beta: G \Rightarrow H$ and get a natural transformation $\alpha\beta: F \Rightarrow H$. We can also define identity natural transformations. Again, it's not hard to check the left and right unit law and associativity for these.

Definition 4. *Given functors $F, G: C \to D$, a* **natural isomorphism** *$\alpha: F \Rightarrow G$ is a natural transformation that has an* **inverse**, *i.e. a natural transformation $\beta: G \Rightarrow F$ such that $\alpha\beta = 1_F$ and $\beta\alpha = 1_G$.*

It's not hard to see that a natural transformation $\alpha: F \Rightarrow G$ is a natural isomorphism iff for every object $x \in C$, the morphism α_x is invertible.

Definition 5. *A functor $F: C \to D$ is an* **equivalence** *if it has a* **weak inverse**, *that is, a functor $G: D \to C$ such that there exist natural isomorphisms $\alpha: FG \Rightarrow 1_C$, $\beta: GF \Rightarrow 1_D$.*

Definition 6. *A functor $F: C \to D$ is* **essentially surjective** *if for every object $x \in D$ there is an object $\tilde{x} \in C$ such that $F(\tilde{x}) \cong x$.*

Definition 7. *A functor $F: C \to D$ is* **full** *if for every pair of objects $x, y \in C$, the function $F: \hom(x, y) \to \hom(F(x), F(y))$ is onto.*

Definition 8. *A functor $F: C \to D$ is* **faithful** *if for every pair of objects $x, y \in C$, the function $F: \hom(x, y) \to \hom(F(x), F(y))$ is one-to-one.*

I will be glad to give you further hints if you need them. The fun part is constructing the weak inverse of a functor F using the fact that it's essentially surjective, full and faithful. This is a categorified version of constructing the inverse of a function using the fact that it's surjective and injective.