Categorified Inner Products

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As a warmup for using stuff types to study Feynman diagrams, let's use them to work out inner products of some states of the harmonic oscillator! But first, let me remind you of some things you need to know.

Recall that the **Weyl algebra**, $\mathbf{W}$, *is the associative algebra over* $\mathbb{C}$ *generated by elements* q, p *satisfying this relation:*

$$pq - qp = -i,$$

or

$$[p, q] = -i$$

for short. The Weyl algebra has a number of interesting representations, but for now let us work in the Fock representation.

In the **Fock representation**, *we think of* p *and* q *as operators on the space* $\mathbb{C}[z]$ *consisting of complex polynomial in one variable* z. *To do this, we first define the* **annihilation operator** a *and* **creation operator** a^*, *as follows:*

$$(a\psi)(z) = \frac{d\psi}{dz}(z), \qquad\qquad (a^*\psi)(z) = z\psi(z).$$

These satisfy

$$[a, a^*] = 1.$$

Then we define the **position operator** q *and* **momentum operator** p *in terms of these:*

$$q = \frac{a + a^*}{\sqrt{2}}, \qquad\qquad p = \frac{a - a^*}{\sqrt{2}i}.$$

These satisfy $[p, q] = -i$, *as desired.*

In the homework k**-Colorings as Categorified Coherent States**, *we saw that* $\mathbb{C}[z]$ *has a unique inner product such that* $1 \in \mathbb{C}[z]$ *is a unit vector and*

$$\langle a^*\phi, \psi\rangle = \langle \phi, a\psi\rangle.$$

This inner product is given explicitly by

$$\langle z^n, z^m\rangle = n!\, \delta_{nm}.$$

We can complete $\mathbb{C}[z]$ *in the norm associated to this Hilbert space, obtaining a Hilbert space called the* **Fock space**. *If we call this* $\mathbf{K}$, *we have*

$$\mathbb{C}[z] \subset \mathbf{K} \subset \mathbb{C}[[z]]$$

since

$$\mathbf{K} = \{\sum_{n\in\mathbb{N}} a_n z^n \colon \sum_{n\in\mathbb{N}} n!\, |a_n|^2 < \infty\}.$$

Elements of $\mathbf{W}$ *act as densely defined unbounded operators on* $\mathbf{K}$.

The vector z^n has an important physical significance! When we normalize it, we get a state in which the quantum harmonic oscillator has n quanta of energy. More precisely, if we define the **harmonic oscillator Hamiltonian** *by*

$$H_0 = \frac{1}{2}(p^2 + q^2 - 1) = a^* a,$$

then we have

$$H_0 z^n = n z^n.$$

The state $z^0 = 1$ is especially important since it's the **ground state** *of the harmonic oscillator: that is, the state with the least energy.*

The vector $e^{kz} \in \mathbf{K}$ for $k \in \mathbb{C}$ also has an important significance! When we normalize it, we get a **coherent state***: a state $\psi \in \mathbf{K}$ for which the uncertainty in position:*

$$\Delta_\psi(q) = \langle \psi, (q - \langle \psi, q\psi \rangle)^2 \, \psi \rangle$$

equals the uncertainty in momentum:

$$\Delta_\psi(p) = \langle \psi, (p - \langle \psi, p\psi \rangle)^2 \, \psi \rangle$$

while making their product as small as allowed by uncertainty principle:

$$\Delta_\psi(p) \, \Delta_\psi(q) = \frac{1}{2}.$$

Okay — now let's categorify some of this stuff! Suppose K is a tame groupoid with cardinality k. Recall these stuff types:

- *Z^n = "being a totally ordered n-element set"*
- *$Z^n /\!\!/ n!$ = "being an n-element set"*
- *E^{KZ} = "being a K-colored finite set"*

Using the inner product of stuff types, we have seen in class that

$$\langle Z^n, Z^n \rangle \sim n!.$$

(Often we use $n!$ to stand for the group of all permutations of the n-element set; here it stands for the underlying set *of this group.) Decategorifying, this implies that*

$$\begin{array}{rcl} \langle z^n, z^n \rangle & = & \langle |Z^n| \, , \, |Z^n| \rangle \\ & = & |\langle Z^n, Z^n \rangle| \\ & = & |n!| \\ & = & n! \end{array}$$

(At the end here, $n!$ stands for the number *of elements in the set we've been calling $n!$.)*

Now, let's work out some more inner products using similar reasoning!

1. Show that

$$\langle Z^n /\!\!/ n! \, , \, Z^n /\!\!/ n! \rangle \simeq 1 /\!\!/ n!$$

where $1 /\!\!/ n!$ is the groupoid with one object and $n!$ as the group of automorphisms of that object. Conclude that

$$\langle z^n / n! \, , \, z^n / n! \rangle = 1/n!.$$

2. Show that

$$\langle Z^n \, , \, Z^n /\!\!/ n! \rangle \simeq 1$$

where 1 is the one-element set. Conclude that

$$\langle z^n \, , \, z^n / n! \rangle = 1.$$

3. Show that

$$\langle Z^n , E^{KZ} \rangle \simeq K^n$$

where K^n is the product of n copies of the groupoid K. Conclude that

$$\langle z^n , e^{kz} \rangle = k^n.$$

4. Show that

$$\langle Z^n /\!\!/ n! \, , \, E^{KZ} \rangle \simeq K^n /\!\!/ n!$$

where $K^n /\!\!/ n!$ is the weak quotient of K^n by the group $n!$. Conclude that

$$\langle z^n / n! \, , \, e^{kz} \rangle = k^n / n!$$

Hint: We have discussed the weak quotient $K^n /\!\!/ n!$ in class — do you remember our intuitive description of it?

5. Show that

$$\langle E^{KZ} , E^{KZ} \rangle \sim E^{K^2}$$

where E^{K^2} is the result of evaluating the stuff type E^Z at the groupoid K^2. Conclude that

$$\langle e^{kz} , e^{kz} \rangle = e^{k^2}.$$

*You proved this last equation already in k-***Colorings as Categorified Coherent States***; now you have categorified it! You couldn't do this without stuff types....*