Math 260: Categorified inner products

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We will consistently use skeleta. The skeleton of FinSet_0 is E, the category with $\mathbf{N}$ as its set of objects, no morphism other that automorphisms, and $\mathrm{Aut}(n) = \mathbf{Z}_n$.

1. $\langle Z^n/\!\!/n!, Z^n/\!\!/n!\rangle$.

The structure type $Z^n/\!\!/n!$, being an n-element set, has for its total groupoid the groupoid of n-element sets with bijections, which is a full subcategory of E, and whose skeleton consists of an n-element set as its single object and the group $n!$ of permutations of its elements as morphisms. In other words,

$$Z^n/\!\!/n!\colon 1/\!\!/n! \to E$$

is the inclusion. Now, the pull-back

$$\begin{tikzcd} \langle Z^n/\!\!/n!, Z^n/\!\!/n!\rangle \arrow[r] \arrow[d] & 1/\!\!/n! \arrow[d, "Z^n/\!\!/n!"] \\ 1/\!\!/n! \arrow[r, "Z^n/\!\!/n!"] & E \end{tikzcd}$$

has objects of the form

$$n^\bullet \xrightarrow{\alpha} n^\bullet\ ,$$

where $n \in E$, $\bullet \in 1/\!\!/n!$ and $\alpha\colon n \to n$. Morphisms are pairs of morphisms

$$\begin{tikzcd} \bullet \arrow[d, "\gamma"] & \bullet \arrow[d, "\gamma'"] \\ \bullet & \bullet \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} n^\bullet \arrow[r, "\alpha"] \arrow[d, "(Z^n/\!\!/n!)(\gamma)"'] & n^\bullet \arrow[d, "(Z^n/\!\!/n!)(\gamma')"] \\ n^\bullet \arrow[r, "\beta"] & n^\bullet \end{tikzcd} \quad \text{commutes,}$$

where $\gamma\colon \bullet \to \bullet$ can be chosen freely, and determines γ' uniquely. Hence, all $n^\bullet \xrightarrow{\alpha} n^\bullet$ are isomorphic and have automorphism groups isomorphic to $n!$. It follows that

$$\langle Z^n/\!\!/n!, Z^n/\!\!/n\rangle \simeq 1/\!\!/n!.$$

2. $\langle Z^n, Z^n/\!\!/n!\rangle$.

The skeleton of the groupoid of totally-ordered n-element sets with order-preserving bijections consists of a single n-element set as object, and the identity permutation as morphism, since no other permutations preserve a total order. This is equivalent to the groupoid 1.

The structure type "being a totally-ordered n-element set" is the inclusion functor $Z^n\colon 1 \to E$ and the pull-back

$$\begin{tikzcd} \langle Z^n, Z^n/\!\!/n!\rangle \arrow[r] \arrow[d] & 1/\!\!/n! \arrow[d, "Z^n/\!\!/n!"] \\ 1 \arrow[r, "Z^n"] & E \end{tikzcd}$$

again has objects of the form

$$n^\bullet \xrightarrow{\alpha} n^\bullet \qquad \text{with} \quad \alpha \in E,$$

with morphisms being of the form

$$\begin{tikzcd} \bullet \arrow[d, "1"] & \bullet \arrow[d, "\gamma"] \\ \bullet & \bullet \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} n^\bullet \arrow[r, "\alpha"] \arrow[d, "Z^n(1)"'] & n^\bullet \arrow[d, "(Z^n/\!\!/n!)(\gamma)"] \\ n^\bullet \arrow[r, "\beta"] & n^\bullet \end{tikzcd} \quad \text{commutes,}$$

so $(Z^n /\!\!/ n!)(\gamma) = \alpha^{-1}\beta$ is determined uniquely by α and β and, since $Z^n /\!\!/ n!$ is faithful, there is a unique such γ in $\mathbf{Z}^n /\!\!/ n!$ for each pair α, β. Hence, all $n^\bullet \xrightarrow{\alpha} n^\bullet$ are isomorphic in exactly one way, and

$$\langle Z^n, Z^n /\!\!/ n! \rangle \simeq 1.$$

3. $\langle Z^n, E^{KZ} \rangle$.

The proper generalization of K-colouring and colour-preserving bijections (when K is a set) to the case when K is a groupoid is "K-flavourings with flavour-preserving, colour-changing bijections" (flavour-compatible bijections for short). To see why, consider a skeleton of K, whose only morphisms are automorphisms. We can visualize the skeleton as a collection of "flavoured objects" each of which with "internal (colour) degrees of freedom", and "colour-changing, flavour-preserving morphisms". This sounds a lot like quantum chromodynamics!

The stuff type E^{KZ}, "being a K-flavoured finite set", has for its total groupoid the groupoid of K-flavoured finite sets with flavouring-compatible bijections, denoted E^K. In symbols,

$$E^{KZ}\colon E^K \to E.$$

To fix the notation, an object of E^K is an n-tuple $k = (k_1, \ldots, k_n)$ of objects of K, for some $n \in E$; and a morphism $\kappa\colon k \Rightarrow k'$, where $k' = (k_1, \ldots, k_n)$ for the same n, consists of a bijection $\kappa_0\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ and morphisms $\kappa_i\colon k_i \to k'_{\alpha_0(i)}$. In other words, k, k' are objects of K^n and $(\kappa_1, \ldots, \kappa_n)$ is a morphism of K^n. The stuff type $E^{KZ}\colon E^K \to E$ maps $E^{KZ}(k) = \{1, \ldots, n\}$ and $E^{KZ}(\kappa) = \kappa_0$.

The pull-back

$$\begin{tikzcd} \langle Z^n, E^{KZ} \rangle \arrow[r] \arrow[d] & E^K \arrow[d, "E^{KZ}"] \\ 1 \arrow[r, "Z^n"] & E \end{tikzcd}$$

has objects of the form

$$n^\bullet \xrightarrow{\alpha} n^k, \qquad \text{with} \quad \alpha \in n! \qquad \text{and} \quad k \in K^n.$$

and morphisms of the form

$$\begin{tikzcd} \bullet \arrow[d, "1"] & k \arrow[d, "\kappa"] \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} n^\bullet \arrow[r, "\alpha"] \arrow[d, "Z^n(1)"'] & n^k \arrow[d, "E^{KZ}(\kappa)"] \\ n^\bullet \arrow[r, "\beta"] & n^{k'} \end{tikzcd} \quad \text{commutes,}$$

that is, $\beta = \alpha\kappa_0$. Given α and β, κ_0 is uniquely determined, and κ is a morphism of K^n such that $\kappa_0 = \alpha^{-1}\beta$. There may be any number (including none) of κ with that κ_0. Since there are no other restrictions on k or κ, it is clear that

$$\langle Z^n, e^{KZ} \rangle \simeq K^n.$$

4. $\langle Z^n /\!\!/ n!, e^{KZ} \rangle$.

By definition of X_n as the pull-back

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & X \arrow[d, "F"] \\ 1 /\!\!/ n! \arrow[r, "Z^n /\!\!/ n!"] & E \end{tikzcd}$$

we have that

$$\langle Z^n /\!\!/ n!, F \rangle = X_n.$$

so

$$\langle Z^n /\!\!/ n!, e^{KZ} \rangle \simeq K^n /\!\!/ n!.$$

5. $\langle e^{KZ}, e^{KZ}\rangle$.

The pull-back

$$\begin{tikzcd} \langle E^{KZ}, E^{K}Z\rangle \arrow[r] \arrow[d] & E^{K} \arrow[d, "E^{KZ}"] \\ E^{K} \arrow[r, "E^{KZ}"] & E \end{tikzcd}$$

has objects of the form

$$n^k \xrightarrow{\alpha} n^{k'}\,, \qquad \text{with} \quad \alpha \in n! \qquad \text{and} \quad k, k' \in K^n,$$

and morphisms of the form

$$\begin{tikzcd} k \arrow[d, "\kappa"] & k' \arrow[d, "\kappa'"] \\ k'' & k''' \end{tikzcd} \quad \text{such that} \quad \begin{tikzcd} n^k \arrow[r, "\alpha"] \arrow[d, "E^{KZ}(\kappa)"'] & n^{k'} \arrow[d, "E^{KZ}(\kappa')"] \\ n^{k''} \arrow[r, "\alpha'"] & n^{k'''} \end{tikzcd} \quad \text{commutes.}$$

In other words, $\alpha\kappa_0' = \kappa_0\alpha'$. In this case, once κ_0 is (freely) specified, then κ_0' is determined. Other than that, as long as κ and κ' preserve the flavouring, they can be freely specified, as can k, k'. This corresponds to a K^2-flavouring of the n-element set, with morphisms $\kappa = (\kappa_0, \kappa', \kappa'')$, where $\kappa_0 \colon n \to n$ is a flavour-preserving permutation, and κ', κ'' are colour-changing morphisms in K^n. Hence, κ is a morphism of $(K^2)^n /\!\!/ n!$, and

$$\langle E^{KZ}, E^{KZ}\rangle \simeq E^{K^2}.$$