A pedagogical analogy between forgetful functors and certain polynomials

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Introduction

This much, hopefully, is clear enough: Let C and D be groupoids, and let F be a functor from C to D. Then F is **faithful** if, given objects x and y of C and morphisms f and g from x to y in C, whenever $F(f)$ and $F(g)$ are equal in D, then $f = g$ back in C.

$$\begin{tikzcd} \forall\ x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & y\ , \end{tikzcd} \quad \begin{tikzcd} F(x) \arrow[r, bend left, "F(f)"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Vert"] & F(y) \end{tikzcd}$$

$$\begin{tikzcd} \Rightarrow\ x \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Vert"] & y \end{tikzcd}$$

Also, F is **full** if, given objects x and y of C, whenever h is a morphism from $F(x)$ to $F(y)$ in D, then there is a morphism g from x to y back in C such that $h = F(g)$ in D.

$$\begin{tikzcd} \forall\ x & y\ , \end{tikzcd} \quad \begin{tikzcd} \forall\ F(x) \arrow[r, bend left, "h"] & F(y)\ , \end{tikzcd}$$

$$\begin{tikzcd} \exists\ x \arrow[r, bend right, "g"'] & y\ , \end{tikzcd} \quad \begin{tikzcd} F(x) \arrow[r, bend left, "h"] \arrow[r, bend right, "F(g)"'] \arrow[r, phantom, "\Vert"] & F(y) \end{tikzcd}$$

Finally, F is **essentially surjective** if, whenever z is an object of D, then there is an object y back in C such that there's a morphism h from z to $F(y)$ in D.

$$\forall\ z \qquad ,$$

$$\exists\ \qquad y\ , \quad \exists\ z \xrightarrow{h} F(y)$$

That much, hopefully, is clear enough; and it fits the only obvious pattern.

But now we can define some subsidiary notions: First, F **forgets nothing** if it's essentially surjective, full, and faithful. Next, F **forgets only property** if it's full and faithful. Next, F **forgets at most structure** if it's faithful. Finally, F **forgets at most stuff** always. These form a pattern as well, but it may not be at all clear *why* this is a relevant pattern. It also may not be clear why these are reasonable *names*; what do they say about the nature of stuff, structure, and property *themselves*?

To answer these questions, I put forward an analogy with certain polynomials.

The analogue

Consider a polynomial P of the form $ax^2 + bx + c$, where (to be definite) a, b, and c are natural numbers. We're all familiar with several useful classes of polynomials: First, P is **zero** if a, b, and c are all zero. Next, P is **constant** if a and b are both zero. Next, P is **linear** if a is zero but b is not. Finally, P is **quadratic** if a is not zero.

Actually, the concepts of linear and quadractic polynomials are somewhat unnatural. They're useful notions in the context of finding the *roots* of polynomials, because we need something nonzero to divide by. (Think of a's place in the quadratic formula, for example.) But they're not quite the right *sets* of polynomials. (They're not closed under addition, for example.) So to correct this, let P be **at most linear** if a is zero, and let P be **at most quadratic** always. (Thus the 'certain polynomials' in the title are in fact the at-most-quadratic polynomials.)

These should still be familiar kinds of polynomials, and you can see why the definitions should go in this order, rather than the other order; it's because c really must be the end of the polynomial, whereas a (despite getting its name from the beginning of the alphabet by tradition) is only the beginning of the polynomial because we restricted attention to at-most-quadratic polynomials. Clearly, the concept can be generalised to polynomials of higher degree.

The analogy

Here is the basic idea: In the polynomial $a\mathrm{x}^2 + b\mathrm{x} + c$, the coefficient a represents how unfaithful a functor is, while the coefficient b represents how unfull it is, and the coeffecient c represents how essentially unsurjective it is. Thus the zero polynomial corresponds to an equivalence, a constant polynomial corresponds to a full and faithful functor, an at-most-linear polynomial corresponds to a faithful functor, and an at-most quadratic polynomial corresponds to an arbitrary functor. But these are precisely the functors that we said forgot, respectively, nothing, only property, at most structure, and at most stuff. So these would seem to be sensible notions after all.

And what about polynomials of higher degree? They are generalisations of at-most-quadratic polynomials, and they correspond to generalisations of the notion of functor. Just as there is no reason to stop at x^2, so there is no reason (in principle) to stop at groupoids. We can go on to 2-groupoids, 3-groupoids, and the like; and similarly, we can generalise stuff types to 2-stuff types (which Jim calls 'eka-stuff types'), 3-stuff types, and the like. So the best thing to learn from the analogy with polynomials is to realise that we've barely started!

Homogenous polynomials

There is another way to break down the class of polynomials. This is given by the notion of homogeneous polynomial, a polynomial each of whose terms has the same degree. Since our polynomials are in a single variable, this means that we're basically talking about monomials — except that the zero polynomial is also included.

The first type of homogeneous polynomial is the constant polynomial again, one where both a and b are zero. But the next kind is the **homogeneous linear polynomial**, where a is zero (so it is at most linear) but c is also zero. Finally, we have the **homogenous quadratic polynomial**, where b and c are both zero. Notice that the zero polynomial belongs to each of these groups, but they are otherwise disjoint. And a generic polynomial won't be homogeneous of *any* degree.

The corresponding types of functors are also interesting. A functor that is both full and faithful forgets only property, as before. But now, a functor that is both essentially surjective and faithful **forgets purely structure**. And a functor that is both essentially surjective and full **forgets purely stuff**. An equivalence is all of these; a generic functor is none of these.

Now, a constant polynomial can be written as c, which is a single natural number; metaphorically, we can say that c is the number of properties that the functor forgets.* Similarly, a homogenous linear polynomial is $b\mathrm{x}$, and we can say (even more metaphorically) that the number b measures how much pure structure is forgotten. Finally, a homogenous quadratic polynomial has the form $a\mathrm{x}^2$, and we can say that a measures how much pure stuff has been forgotten.

There are limits to this analogy. An at-most-quadratic polynomial may be specified by giving the numbers a, b, and c *independently*, but this won't work for a functor. You can't simply say, ‹This stuff is forgotten, also this property; and by the way, we also forget this structure here.›. This is because they are tied together; you only know what properties are avaiable *after* the stuff and structure have been specified. Conversely, if you forget a structure, then the properites referring to it (if they can't be reinterpreted in terms of surviving structures) become irrelevant. So unlike with polynomials, the 3 levels are not independent.

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* You cannot push this metaphor too far; any property usually has infinitely many derived properties that are forgotten as well, and the same set of derived properties may be generated from various finite subsets of different sizes. As a value of c, only zero has a precise meaning in the analogy.

Addition and composition

Nevertheless, there is a real and important sense in which an arbitrary functor may be built up in 3 steps, just as an arbitrary at-most-quadratic polynomial may be built up in 3 terms. One just needs to be careful about the *order*. To deal with this, I must extend the analogy to cover polynomial addition. Specifically, the addition of polynomials is analogous to the composition of functors. The idea to keep in mind here is that just as the coefficients of $P + Q$ are as big as the coefficients of P and Q combined, so the functor $F \circ G$ forgets as much as F and G forget combined.

One relevant property of the natural numbers is that $0 + 0 = 0$. Thus a sum of at-most-linear polynomials (meaning that a is zero) is at most linear; similarly, a composition of forgetting-at-most-structure functors forgets at most structure (meaning that it's faithful). The same goes for the full functors, the forgetting-purely-stuff functors, the forgetting-only-property functors, and so on. Despite the warning in the footnote, there is a good reason that I defined my polynomials over the natural numbers. Recall this fact about natural numbers: If $a + b$ is zero, then a and b are also both zero. A consequence for polynomials is that if $P + Q$ is zero, then P and Q are also both zero. This corresponds to the fact for functors that if $F \circ G$ is an equivalence, then F and G are also both equivalences.[†]

Functor factorisation

Now I can tackle the factorisation of functors into functors that forget purely one level, just as polynomials can be broken down into (homogeneous) monomials. Given groupoids C and D and a functor $F\colon C \to D$, we can break down F as a composition, *first* of a functor that forgets purely stuff, *then* a functor that forgets purely structure, and *finally* a functor that forgets only property. (The order of composition is important!)

$$C \xrightarrow{F|\,\mathrm{Mor}\,C} \underline{\mathrm{coim}}\,F \xrightarrow{F|\,\mathrm{Ob}\,C} \overline{\mathrm{im}}\,F \hookrightarrow D$$

First, let $\underline{\mathrm{coim}}\,F$ be the **weak coimage** of F, a category whose objects are the objects of C but whose morphisms come from D; specifically, given objects x and y in C, the morphisms from x to y in $\underline{\mathrm{coim}}\,F$ are those morphisms from $F(x)$ to $F(y)$ in D that lie in the image of F. Then F induces a functor from C to $\underline{\mathrm{coim}}\,F$, which acts like the identity on objects but like F on morphisms. This functor is essentially surjective and full; it forgets purely stuff. Next, let $\overline{\mathrm{im}}\,F$ be the **full image** of F in D; its objects are those objects of D that lie in the image of F, and its morphisms are *all* of the morphisms in D between these objects. Then F induces a functor from $\underline{\mathrm{coim}}\,F$ to $\overline{\mathrm{im}}\,F$, which acts like F on objects but like inclusion on morphisms. This functor is essentially surjective and faithful; it forgets purely structure. Finally, the inclusion of $\overline{\mathrm{im}}\,F$ into D is full and faithful; it forgets only property. (Incidentally, the image of F never shows up itself, since it may not be closed under composition unless F is injective on objects.)

Summary

The analogy is between at-most-quadratic polynomials over the natural numbers and functors between groupoids.

Polynomials	Functors
Zero	Equivalence
Constant	Forgets only property
At most linear	Forgets at most structure
Homogeneous linear	Forgets purely structure
Homogeneous quadratic	Forgets purely stuff
Addition	Composition
Sum of homogeneous polynomials:	Composition of purely forgetful functors:
$P = (ax^2) + (bx) + (c)$	$F\colon C \xrightarrow{F|\,\mathrm{Mor}\,C} \underline{\mathrm{coim}}\,F \xrightarrow{F|\,\mathrm{Ob}\,C} \overline{\mathrm{im}}\,F \hookrightarrow D$

[†] Nevertheless, the warning against trusting the analogy too much still applies. If $P + Q$ is at most linear, then P and Q must also each be at most linear. But if $F \circ G$ forgets at most structure, then F and G need not forget at most structure. This part of the analogy can be fixed by paying careful attention to the order of composition, but it's somewhat complicated and I won't discuss it here.